Deterministic Sampling Methods for Spheres and $SO(3)$

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Abstract

This paper addresses the problem of generating uniform deterministic samples over the spheres and the three-dimensional rotation group, $SO(3)$. The target applications include motion planning, optimization, and verification problems in robotics and in related areas, such as graphics, control theory and computational biology. We introduce an infinite sequence of samples that is shown to achieve: 1) low-dispersion, which aids in the development of resolution complete algorithms, 2) lattice structure, which allows easy neighbor identification that is comparable to what is obtained for a grid in $\mathbb{R}^d$, and 3) incremental quality, which is similar to that obtained by random sampling. The sequence is demonstrated in a sampling-based motion planning algorithm.

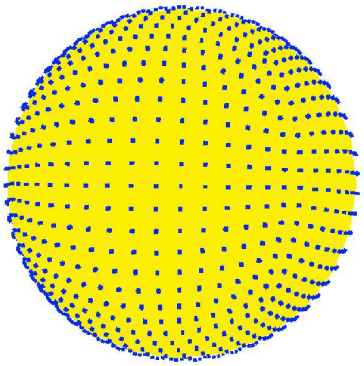


Figure 1: Distribution of points on the sphere S^2 generated by a grid (Sukharev [20]) on each spherical face.

1 Introduction

Many important algorithms developed in robotics and related areas require careful sampling over spheres. In recent years, the paradigm of sampling-based motion planning has led to algorithms that can solve many challenging problems by combining collision detection, search algorithms, and sampling strategies over the configuration space. General sampling over spheres arises in many forms of planning and optimization in which some number of directions are locally explored. For example, some potential field approaches [3, 10] involve sampling local directions to obtain an approximate gradient descent. The exact expression of the gradient may be too costly or even unavailable. One important special case of sampling over spheres is sampling over the 3D rotation group, $SO(3)$, which involves sampling over half of the three-sphere, S^3. One of the main motivations for this paper is the problem of motion planning for a rigid body in $\mathbb{R}^3$.

We are particularly interested in the development of deterministic sampling methods. Although most existing motion planning methods currently use random sampling, they are limited to probabilistic forms of completeness. With deterministic sampling, resolution completeness guarantees are possible. This is particularly valuable in the area of system verification, in which one must guarantee that a system behaves correctly under all possible trajectories. The intractability of most of these problems leads naturally to sampling based approaches. While it may be valuable to verify a system down to some level of resolution, random sampling might leave doubts about whether the space was adequately covered. In some cases, deterministic sampling has even led to practical performance improvements in comparison to random sampling [13, 14, 15]. The techniques presented in the present paper build on recent work to develop uniform, deterministic sampling techniques for motion planning [6, 12, 13].

The particular problem of sampling over spheres presents many unique challenges. The vast majority of sampling literature considers placing points in a unit d-dimensional cube, $[0,1]^d \subset \mathbb{R}^d$ (see [12, 15]). This might correctly capture some configuration spaces that arise in robotics; however, the majority of applications involve other topological spaces, such as $\mathbb{R}P^3$, which arises from rigid body rotations, or toroidal manifolds, which arise from a series of revolute joints of a manipulator. In these cases, special sampling techniques should be developed because quality measures for sets of samples depend on the topology. For example, the maximum distance that a configuration could be from its nearest sample depends on the metric, which is induced partly by the topology.

In addition to topological issues, the way that a configuration space is parameterized is of critical importance to defining notions of uniformity. A collection of samples that are uniform with respect to one parameterization of the configuration space might seem extremely biased using another parameterization. It might seem that there is no way to avoid this frustrating issue, but fortunately for the case of $SO(3)$, there is an intrinsic notion of uniformity that is given by the Haar measure [7] (this will be defined in Section 2). Using this notion, the natural parameterization of $SO(3)$ is the set of unit quaternions (with antipodal identification), and our sampling methods will be developed to achieve rigorous notions of uniformity in this case.

To maximize the potential for impact on motion planning and related areas, our goal has been to develop a sampling method that achieves 1) uniformity, 2) lattice structure, and 3) incremental quality. Uniformity means good covering of the space is obtained without unwanted bias, clumping or gaps. This can be formulated in terms of optimizing discrepancy or dispersion [14, 15, 4]. The uniformity notion considered here is actually more "uniform" than what is obtained by random sampling. Lattice structure means that for every sample, the location of nearby samples can easily be determined as part of a regular pattern (as in neighbors on a grid, for example). Incremental quality means that if the sampling method is considered as an infinite sequence, then the sequence may be truncated after any finite number of samples and good coverage will be obtained. This is an important characteristic of pseudo-random number sequences, making them desirable for many past motion planning algorithms [1, 5, 9, 19, 22]. We would like to obtain the same behavior, even though the sequence is deterministic, uniform, and has lattice structure.

2 Quality Measures for the Distributions of Points on Spheres

We consider generating samples over spheres and $SO(3)$. Let S^d represent a d-dimensional sphere, embedded in $\mathbb{R}^{d+1}$ as

$$S^d = \{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}.$$

The set of all rotations in $\mathbb{R}^3$ is denoted as $SO(3)$, which is defined as the set of all 3×3 orthonormal matrices. It will be helpful to sometimes represent $SO(3)$ as the set, H, of unit quaternions, each of which is expressed as $h = a + bi + cj + dk$, with the identification $h \sim -h$ [11]. Note that it appears that $H = S^3$, except that antipodal points on S^3 are identified in the definition of H. This leads to a close relationship between sampling on sphere and sampling on $SO(3)$.

Now that the spaces have been defined, the next task is to define the quality of samples. Consider sphere S^d over which the finite set of points A is generated.

Definition 2.1 *For a finite point set A generated over the sphere S^d the discrepancy of A with respect to a given family $\mathcal{R}$ of subsets of S^d, called ranges, is defined by*

$$D_{\mathcal{R}}(A) = \sup_{R \in \mathcal{R}} \left| \frac{|A \cap R|}{|A|} - \mu(R) \right|,$$

where μ denotes the rotation invariant measure of the sphere S^d in Euclidean space $\mathbb{R}^{d+1}$, and $|\cdot|$ applied to a finite set denotes its cardinality.

In the case of $SO(3)$ the measure defined on S^3 as above corresponds to the Haar measure defined over the set of all rotation matrices.

The range spaces that are usually considered on the sphere are the set of spherical caps, i.e., intersections of the sphere with half spaces; or the set of spherical slices, i.e., intersections of two half-spheres [4, 16].

Definition 2.2 *The dispersion of a finite set A is defined by*

$$d_{\mathcal{R}}(A) = \sup_{q \in S^d} \min_{p \in A} \rho(q, p),$$

in which ρ is a rotation invariant metric over S^d.

Having these definitions of uniformity in mind, in what follows we propose a general approach to sampling on spheres and $SO(3)$. As a particular example we show how to generate a low-dispersion and low-discrepancy sample set which has additional useful

properties: it is incremental, has lattice structure, and it can be efficiently generated. We show how these samples can be applied to the problems of motion planning.

3 Exploiting the Regularity of Platonic Solids

Our general approach to sampling is based on Platonic solids. In $\mathbb{R}^3$, a Platonic solid or *regular polyhedron*, is a polyhedron for which every face is a copy of a regular polygon, fixed over all faces, and the degree of every vertex is fixed. Let (v, e, f) denote the numbers of vertices, edges, and faces of a regular polyhedron. Although there are an infinite number of regular polygons, there are only five regular polyhedra: tetrahedron (4,6,4), cube (8,12,6), octahedron (6,12,8), icosahedron (12,30,20), and dodecahedron (20,30,12). The notion of regular polyhedron can be generalized to higher dimensions to obtain a *regular polytope*. In $\mathbb{R}^4$, it turns out that there are six regular polytopes: simplex (5,10,10,5), cube (16,32,24,8), cross polytope (8,24,32,16), 24 cell (24,96,96,24), 120 cell (600,1200,720,120), 600 cell (120,720,1200,600). The forth element in each sequence denotes the number of 3D cells (which are regular polyhedra). Finally, in $\mathbb{R}^d$ for any $d > 4$, there are only three regular polytopes: simplex, cube, and cross polytope.

We first address the problem of generating a uniformly distributed set of points over S^d. Consider inscribing any $(d+1)$-dimensional regular polytope inside of S^d, so that all of its n vertices lie in S^d. The set of vertices are beautifully arranged around S^d so that the points are evenly spaced. Furthermore, the edges of the polytope yield a regular lattice structure that is natural for building roadmaps in planning problems. For the case of sampling $SO(3)$, we simply use a set of vertices that lie in one hemisphere (making sure that no antipodal pairs of points appear in the set). The edges can be obtained directly from the polytope by making the appropriate identification of antipodal pairs.

Unfortunately, there are only a few combinations of n and d, for which these ideal samples may be constructed for S^d and $SO(3)$. This might be suitable for some applications, such as picking a set of candidate directions from S^d for gradient descent of a potential function; however, in general, we would like to a have a nice distribution of points for any value of n.

To the best of our knowledge, it is impossible to perfectly space n points around S^d, for any n and for $d > 1$. One simple idea that increases the number of samples is place one point in the center of each of the c d-cells of some regular polytope, and lift it to S^d. If we take the union of these points with the set of v polytope vertices, a nice point set of size $c + v$ may be obtained. If more points are placed; however, the problem becomes more complicated. Therefore, we are willing to tolerate some distortion in the distribution of points. It still seems useful, however, to borrow some of the properties of the regular polytopes to generate good samples. The general idea pursued in this paper is to sample uniformly on the surface of the regular polytope, and then transform generated distribution on the surface of the sphere. We next describe this general method and discuss the induced distortion.

Consider a $(d+1)$-dimensional regular polytope inscribed in the sphere S^d. Suppose there exists a good method of sampling the surface of this polytope. The faces (d-dimensional cells) of the polytope, if projected outward to the surface of the sphere, form a tiling of the surface with the d-dimensional *spherical polytopes*. Consider some particular face, F, and its corresponding spherical face, F'. Each point inside F can be described by the barycentric coordinate systems induced by vertices of F after its triangulation. Now imagine that a distribution of points is generated inside F. Each of the points in this distribution can be obtained through several steps of linear interpolation between the vertices of the barycentric coordinate systems. The distribution on F' can be obtained then through similar steps of interpolating between the vertices of F', except that the interpolation should be done on the surface of the sphere [17]. This idea is similar to the one proposed in [2] for stratified sampling of spherical triangles. As an example, consider a cube inscribed in the sphere S^2, and sample the surface of the cube by putting the Sukharev grid [12, 20] on each square face. Using the proposed method we get a distribution of samples on S^2 as shown on Figure 1.

The distribution of points on the sphere S^d obtained by this method will introduce distortion since spherical arcs corresponding to the intervals inside F with the same length may have different lengths in F'. The amount of the distortion, and therefore bounds on the dispersion and discrepancy, can be obtained through the analysis of the maximal arc differences.

This idea can also be adapted to $SO(3)$ (and in general to the projective space of any dimension). Take a four-dimensional regular polytope inscribed in S^3 and use only half of the faces to generate the distribution

on the surface. We pick the faces so that in the set of used faces, there must not exist a pair of antipodal points, one from each of two different faces. This way the obtained samples will cover exactly half of the sphere, which forms $SO(3)$ surface.

Next we show how to generate a layered Sukharev grid sequence on S^d based on the inscribed cube and the bounds on the dispersion and the discrepancy of this sequence.

4 A Sample Sequence Based on Cubes

In this section first we make an overview of the techniques existing for sampling unit cubes. Next we show a particular sequence adapted to the spheres using the proposed general method and we analyze the uniformity properties of this sequence.

4.1 Sampling in Cubes

The subject of uniform sampling inside unit cube $[0,1]^d$ has been studied extensively for decades (see [14, 15]). Here are some brief concepts.

There are two main sampling families that are considered in the literature: *point sets* and *sequences*.

For a *point set*, the number of points, n, that should be placed in the set is specified in advance, and a set of n points is chosen so that the sampling criterion (dispersion or discrepancy) is optimized. The notion of ordering between the points is not defined for the point sets. As an example we could consider the point sets generated by classical grid and Sukharev grid [20] of resolution l in $[0,1]^d$. Each of these sets contains l points per axis and l^d points total. The difference between them is in the way each of these grids places its points in each of the l^d subregions of the cube. Classical grid places a vertex in the origin of each region, whereas Sukharev grid places a vertex at the center of each region. It was proven that the Sukharev grid optimizes the l^∞ dispersion over all of the point sets of size l^d [15, 20]. Classical grid has low dispersion but is not dispersion optimal.

For *sequences* the ordering of the points becomes important. Each next point in a sequence should be chosen so that the sampling criterion is optimized. Sequences are particularly suitable for the motion planning algorithms, where the number of points needed to solve a given problem is not known in advance.

When designing sequences that optimize dispersion, it is useful to consider *multiresolution grid sequences* [13]. A multiresolution grid of resolution l is a grid with 2^l points per axis and 2^{dl} points total. From this definition it follows that a grid of resolution l contains all of the points from resolution $l-1$. The natural way to make this grid incremental is to build it one resolution at a time. During construction of the points from the same resolution level, the recursive procedure at each step adds those points that maximally decrease the discrepancy of the sequence, which extends van der Corput's one-dimensional sequence [21].

As an example, consider a square, $[0,1]^2$, with four grid points inside. The best order of placing these points is: (0, 0), (0.5, 0.5), (0, 0.5), (0.5, 0). To add the next 12 points from resolution 3, what point should be placed first, second, and third out of this sequence? The idea is that every four points should follow the same ordering of quadrants as the first four points (i.e., the first point should fall into the left-bottom rectangle, the next into right-top, and so on). Where exactly the point should be placed within the left-bottom rectangle should be decided by the same criterion that was used to place the first 4 points. In this case the next point is (0.25, 0.25).

The resulting sequence has several important properties: it is incremental, it has low dispersion at each resolution level, it has optimal discrepancy with respect to the set of *canonical rectangles*, it has lattice structure, and there are efficient methods for generating the sequence and performing nearest neighbor queries on it [13]. This makes multiresolution grid sequences particularly useful for motion planning applications.

We will be using a *layered* version of this sequence. A *layered Sukharev grid* of resolution l is a point set containing all the points of Sukharev grids of resolutions $1, 2, 4, ...2^l$. It follows that this grid has $n = \sum_{i=0}^{l} (2^i)^d = (2^{d(l+1)} - 1)/(2^d - 1)$ points total.

A *layered Sukharev grid sequence* builds one Sukharev grid of resolution 2^i at a time, $i = 1, 2, ...$ Points from each of these grids then are generated by the same procedure as for building multiresolution grid sequences.

In what follows we generalize layered Sukharev grid sequence to the sphere S^d. We first show how the points should be generated in each of the spherical cubes, and then how all these points can be combined into one sequence on the sphere.

4.2 Layered Sukharev Grid Sequence for a Spherical Cube

Consider a face, F, of a $(d+1)$-cube inscribed in a sphere S^d. F is a d-dimensional cube, which in each of its corners has d edges. If we project all of these

edges onto the surface of the sphere they form arcs, which delineate a spherical d-cube, F'. The lengths, α, of these arcs are equal for all edges of F. If we consider those equatorial angles that correspond to the edges coming from a common vertex of F, we can define an *angular coordinate system* for the spherical face F'. Indeed, the coordinates $(x_1, x_2, ...x_{n-1})$ with all possible values $x_i \in [0, \alpha]$ specify all possible points of F'.

The construction of the sequence, T, essentially follows the construction of the layered Sukharev grid sequence for the unit cube, except that instead of the Euclidean coordinate system we use the angular coordinate system defined above.

To analyze the dispersion and discrepancy of this sequence we need several definitions. Define the points of the *Sukharev spherical grid* of resolution 2^l as follows:

$$P_l^d = \left\{ \left(\frac{i_1\alpha}{2^l} + \frac{1}{2^{l+1}}, \frac{i_2\alpha}{2^l} + \frac{1}{2^{l+1}}, ..., \frac{i_d\alpha}{2^l} + \frac{1}{2^{l+1}} \right) : i \in \mathbb{Z}, 0 \le i \le 2^l - 1 \right\}$$

Next we define the set of spherical canonical rectangles, which is an extension to the canonical rectangles defined in [13].

Definition 4.1 *Given positive integers d and m, let Q_m^d be the following family of the d-dimensional spherical canonical rectangles:*

$$Q_m^d = \left\{ \left[\frac{i_1\alpha}{2^m}, \frac{(i_1+j_1)\alpha}{2^m} \right) \times ... \times \left[\frac{i_d\alpha}{2^m}, \frac{(i_d+j_d)\alpha}{2^m} \right) : i, j \in \mathbb{Z}, 0 \le i \le 2^m - 1, 1 \le j \le \min(2^m - i, 2) \right\}$$

The following results can be stated about the dispersion and discrepancy of T.

Proposition 4.2 *The dispersion of the sequence T at the resolution level, l, is*

$$d_\rho(T) \le \frac{2\pi}{\sqrt[d]{n(2^d - 1) + 1}}$$

Proof: The largest spherical cap which does not contain any of the points in T will be smaller than the spherical cap with the center at $(\alpha/2, \alpha/2, ..., \alpha/2)$ and the spherical radius $\pi/2^l$. Since $2^l = \left(\sqrt[d]{n(2^d - 1) + 1} \right)/2$ we have that the dispersion is not bigger than $\pi/2^l = 2\pi/\left(\sqrt[d]{n(2^d - 1) + 1} \right)$. ∎

Proposition 4.3 *The relationship between the discrepancy of the sequence T at the resolution level, l, taken over $\widetilde{Q}_l^d = \bigcup_{m=0}^{l} Q_m^d$ and the discrepancy of the optimal over $\widetilde{Q}_l^d$ sequence, T_o, is:*

$$D_{\widetilde{Q}_l^d}(T) \le D_{\widetilde{Q}_l^d}(T_o) + (V_{max} - V_{min})$$

Proof: The optimal sequence, T_o, may place the points in some different order than T. The maximal change in discrepancy that may occur in T comparing to T_o is the difference between the maximal, V_{max}, and the minimal, V_{min}, volumes of the spherical canonical rectangles. Therefore, $D_{\widetilde{Q}_l^d}(T) \le D_{\widetilde{Q}_l^d}(T_o) + (V_{max} - V_{min})$ ∎

Proposition 4.4 *The sequence T has the following properties:*

- *The position of the i-th sample in the sequence T can be generated in $O(\log i)$ time.*
- *For any i-th sample any of the $2d$ nearest grid neighbors from the same layer can be found in $O((\log i)/d)$ time.*

Proof: For the i-th sample it takes $O(\log_{2^d} i) = O((\log i)/d)$ to find its resolution level l. Once l is found, the corresponding point in Sukharev grid of resolution 2^l needs to be generated. It was proved in [13] that this takes $O(\log i)$. Therefore, the total running time for generating one point is $O((\log i)/d + \log i) = O(\log i)$.

The layer of the i-th sample is the Sukharev grid of resolution 2^l. Any of the $2d$ nearest grid neighbors from this layer can be found in $O((\log i)/d)$ using the algorithm described in [13]. ∎

In our analysis we essentially ignored all of the points from the layers below the i-th sample layer, since the number of them is not significant. In practice, it may be efficient to use other layers for generating nearest neighbors. Better bounds on dispersion and discrepancy may also be achieved then.

4.3 Layered Sukharev Grid Sequence for S^d

Now, that we have defined a sequence for each of the spherical cubes, we need to define an ordering in which all of the points from those sequences will be placed on the surface of the sphere. One straightforward way to do this is to place one point from each

of the faces' sequences at a time. The order in which each face should be considered is decided from the following considerations.

Let the union of all of the spherical canonical rectangles determine the range space for the whole sphere. Using the criterion of optimizing the discrepancy over the range space, the ordering of the first $2(d+1)$ points for the resolution level 0 of the sphere can be explicitly computed. Hence, from this point on we can assume that we have such an ordering. Therefore, each next set of $2(d+1)$ points from each of the sequences should follow the same ordering, since this will minimize the discrepancy over the range space. This will guarantee that Proposition 4.3 holds for the generated sequence on the sphere.

Our ongoing research is directed on proving that the same result holds for the larger range spaces, i.e., the ones that include combinations of the spherical rectangles from different spherical cubes.

We can state the following result for the dispersion of the sequence, T_s, on the sphere:

Proposition 4.5 *The dispersion of the sequence T_s at the resolution level l containing $n = 2(d+1) \cdot (2^{d(l+1)} - 1)/(2^d - 1)$ points is*

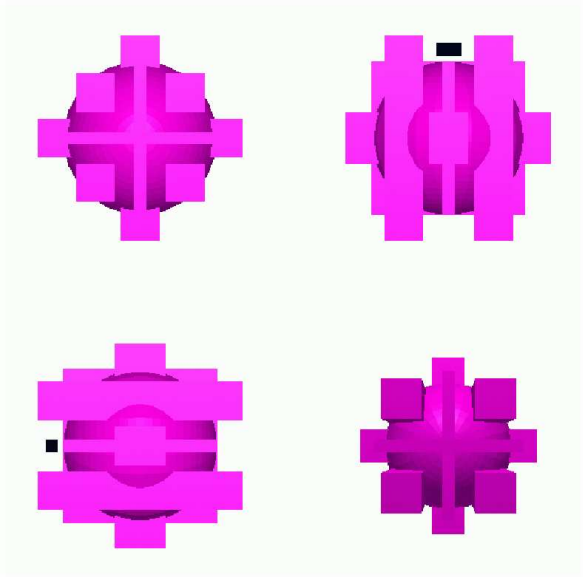
$$d_\rho(T) \leq \frac{2\pi}{\sqrt[d]{\frac{n(2^d-1)}{2(d+1)} + 1}}$$

Proof: Applying the same argument as in the proof of Proposition 4.2, and considering that now $2^l = \left(\sqrt[d]{n(2^d-1)/(2(d+1))+1}\right)/2$, we obtain the desired bound. ∎

5 Experiments

We have implemented our algorithm in C++ and applied to implementations of PRM-based planner [9] in the Motion Strategy Library. The experiments reported here were performed on a 2 Ghz Pentium IV running Linux and compiled under GNU C++.

Performance results are shown in Figures 2, 3. The models that we designed are allowed only to rotate; therefore, the configuration space is $\mathbb{R}P^3$. We compared the number of nodes generated by the basic PRM planner using a pseudo-random sequence of quaternions [18], a pseudo-random sequence of Euler angles, and the layered Sukharev grid sequence. The results for pseudo-random quaternions and Euler angles sequences were averaged over 50 trials. When we tested the deterministic sequence, we made sure



Random Quaternions	Random Euler Angles	Layered Sukharev Grid Sequence
1088	3021	1067

Figure 2: This problem involves moving a robot (black) from the north pole to the south pole. Multiple views of the geometry of the problem are shown (obstacles are drawn in lighter shades) as well as comparisons of the number of nodes generated by different sampling strategies.

that each particular problem does not have any advantage due to coincidental alignment with the grid directions of the sequence. Therefore, in each trial a fixed, random quaternion rotation was premultiplied to each sample, to displace the entire sequence. The results obtained were averaged over 50 trials (a different random rotation was used in each).

Based on our experiments we have observed that the performance of the deterministic sequence is equivalent to the performance of the random sequence for the PRM-based planner, which makes it an alternative approach to random sampling. It is important to note, however, that for some applications, such as verification problem, only deterministic guarantees are acceptable, making random sequences not appropriate.

The results we obtained for the problem in Figure 3 using Euler angles emphasizes the importance of using quaternions and sampling in a way that respects the Haar measure. This problem was never solved using the random Euler angles. The experiment was running for several days, generated 80000 nodes,

Random Quaternions	Random Euler Angles	Layered Sukharev Grid Sequence
909	> 80000	1013

Figure 3: In this example the goal is to move a robot along the corridor. Comparisons of the number of nodes generated by different sampling strategies are shown.

but never found the solution. It is generally known that Euler angle parameterization has its drawbacks, such as gimbal lock and interpolation problems. However, in motion planning, it has been a popular way to parameterize rotations. This example demonstrates the inadequateness of Euler angles parameterization. The interpolation method, ignoring the dependence between the three rotations (yaw-pitch-roll), tries to rotate around three axes simultaneously. In the configuration space with the narrow corridor this results only in those configurations that are in collision.

6 Conclusions

We have proposed a general framework for performing deterministic uniform sampling over spheres and $SO(3)$. We have developed and implemented a particular sequence which extends the layered Sukharev grid sequence designed for the unit cube. We have tested the performance of the sequence in PRM-like motion planning algorithms, which demonstrated that this sequence is a useful alternative to a random sampling. This is in addition to the advantages that this sequence has over random sampling, such as deterministic resolution completeness guarantees and the regular lattice structure.

There are many ways to improve the current work. The spherical distortion grows with the size of the polytope faces and with the dimension. One improvement would be to use regular polytopes that have more faces. For example, for the case of $SO(3)$, a 600-face polytope exists (only 300 of them would be used because of antipodal identification). The difficulty is that our current approach would require sampling over a simplex, as opposed to a cube. Another possibility is to cut and unroll the $(d+1)$-dimensional polytope so that all of its d-dimensional faces form a connected subset of $\mathbb{R}^d$. It may then be possible to adapt a sampling method for rectangular subsets of $\mathbb{R}^d$ to S^d by rolling the polytope back up after sampling.

Another important direction of research is to determine how to combine deterministic sampling methods for two spaces into a method over the Cartesian product space. For example, how can a sample sequence developed for $[0,1]^3$ and another developed for $SO(3)$ be combined to yield a good sequence for a six-dimensional configuration space that corresponds to a set of translations and rotations for a 3D rigid body? In the case of random sampling, it is trivial to combine independent random samples; however, for deterministic methods, one must be very careful to avoid degeneracies. This is the reason, for example, why the Halton sequence [8] uses relatively prime integers as the basis for each dimension.

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