# AMIRKABIR WINTER SCHOOL

# Minimalism in Robotics: From Sensing to Filtering to Planning

## PART 2: SENSING

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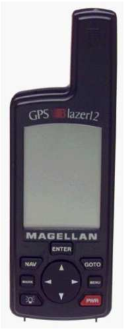
# Physical Sensors

# What Is a Sensor?

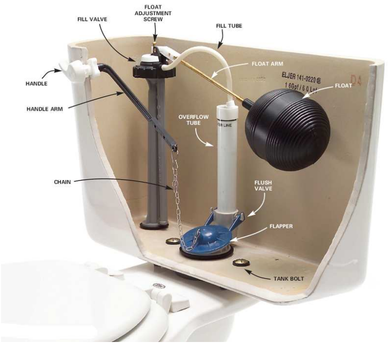


Light-dependent resistor




GPS unit

Wireless card




Toilet float mechanism

We know it when we see it, but will not try to formally classify.

# Where Might We Want to Use Sensors?



Shopping mall

Control room


Assisted living

Coral reef

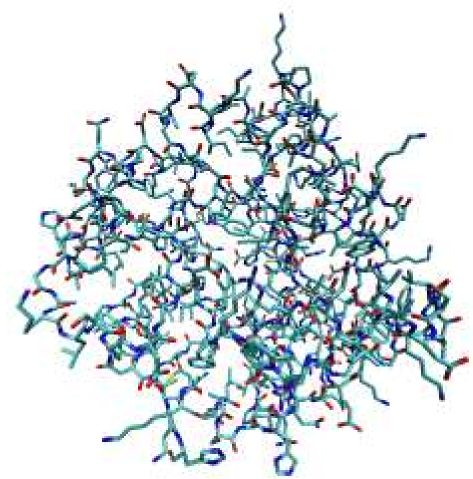
# Where Might We Want to Use Sensors?



Roomba

CMU Boss

UAV



Protein

# What Physical Quantities Are Sensable?



**Spatial:** displacement, velocity, acceleration, distance to something, proximity, position, attitude, area, volume, level/tilt, motion detection

**Temporal:** clock, chronometer (elapsed time), frequency.

**Electromagnetic:** voltage, current, power, charge, capacitance, inductance, magnetic field, light intensity, color. These may operate within a circuit or within open space.

**Mechanical:** solid (mass, weight, density, force, strain, torque), fluid (acoustic, pressure, flow, viscosity), thermal (temperature), calories.

**Other:** chemical (composition, pH, humidity, pollution, ozone), radiation (nuclear), biomedical (blood flow, pressure).

See *CRC Measurement, Instrumentation, and Sensors Handbook*

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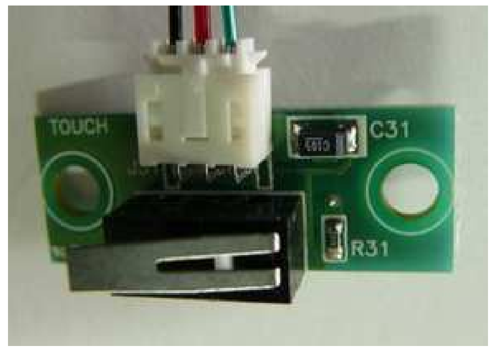
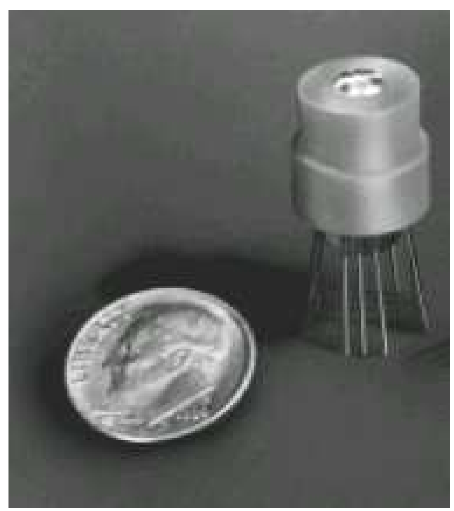
# What Sensors Are Available?






Contact sensor

Sonar

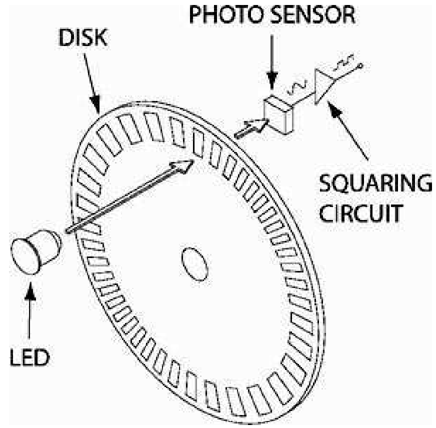


Compass

Microphone

# What Sensors Are Available?






Wheel encoder


Stopwatch/timer

Occupancy detector

Safety beam

# What Sensors Are Available?



Camera

Wii remote

Pressure mat


SICK laser scanner

# Common Sensor Characteristics

- *Transfer function* converts physical phenomenon to sensor reading:
$$g : \mathbb{R} \rightarrow \mathbb{R}.$$

- Domain of $g$ may be *absolute* vs. *relative*.
- $g$ itself may be *linear* or *nonlinear*.
- *Resolution* is given by set of possible $g(x)$.

- *Sensitivity* is set of stimuli that produce same reading.
- *Repeatability* is producing same readings under same phenomena.
- *Calibration* eliminates systematic errors.

You will find these notions in sensor handbooks.

# Physical state spaces

# Physical Sensors vs. Virtual Sensors



**Physical sensor:** The real thing.

**Virtual sensor:** Mathematical model of information obtained from a sensing system.

A virtual sensor could have many alternative physical-sensor implementations.

Identifying which *virtual* sensor is required will lead to better filter design and planning algorithms.

# Physical State Space to Observation Space



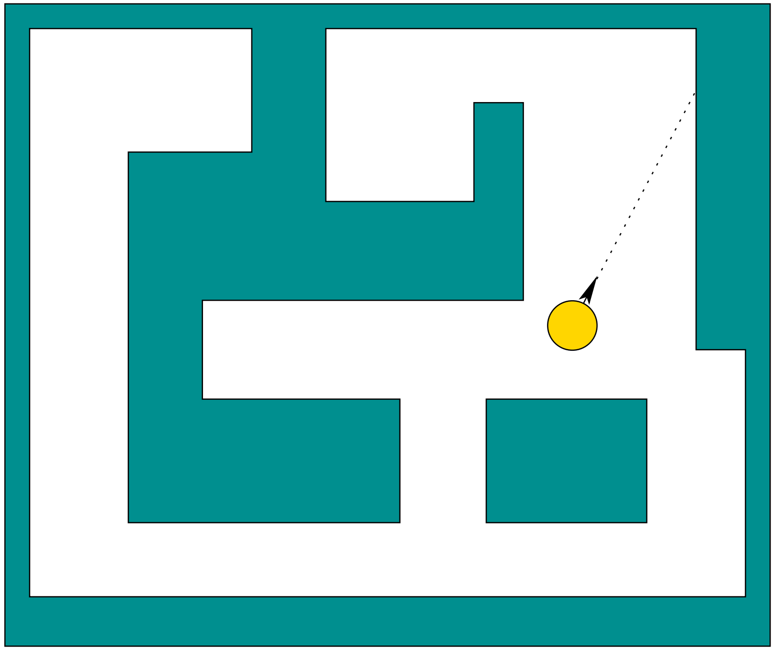
The key idea in this section is to understand how two spaces are related:

1. The *physical state space*, in which each physical state is a cartoon-like description of the possible world external to the sensor.
2. The *observation space*, which is the set of possible sensor output values or observations.

Physical state $\longrightarrow$ a sensor observation

# A Mobile Robot Among Polygonal Walls





- Observation: The wall is $3$ meters away.
- What possible external physical worlds are consistent with that?

# A Common Structure



- Localization only: Set of possible configurations
- Mapping only: Set of possible environments
- Both: Set of configuration-environment pairs

Let $\mathcal{Z}$ be any set of sets.

Each $Z \in \mathcal{Z}$ is a "map" .
Each $z \in Z$ is the configuration or "place" in the map.

Unknown configuration and map yields a state space as:
All $(z, Z)$ such that $z \in Z$ and $Z \in \mathcal{Z}$.

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# State Space For a Planar Mobile Robot



Without any obstacles:

- Any position $(q_x, q_y) \in \mathbb{R}^2$
- Any orientation $q_\theta \in [0, 2\pi)$
- Let *state space* $X$ be all positions and orientations

Can imagine $X \subset \mathbb{R}^3$; however, for orientation, we have additional topology since $q_\theta = 0 = 2\pi$.

Could write $X = \mathbb{R}^2 \times S^1$, in which $S^1$ is a circle and the set of all orientations.

Could write $X = SE(2)$, set of all 2D rigid-body transformations.

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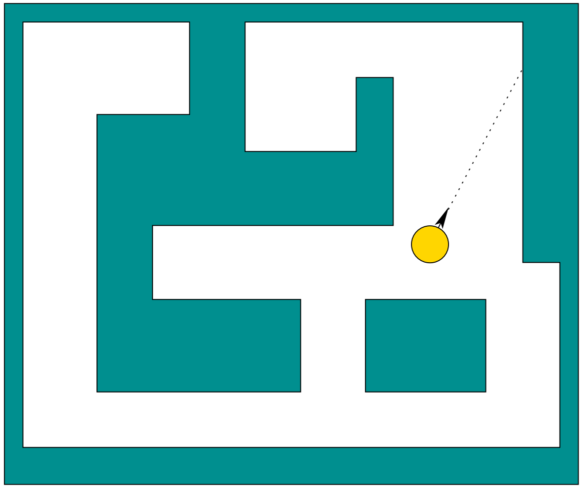

Suppose $E \subset \mathbb{R}^2$ is known to be the set of allowable positions.

Must have $(q_x, q_y) \in E$.

State space: $X = E \times S^1$

# State Space For One of Several Maps



Given a set of $k$ possible maps:

$$\mathcal{E} = \{E_1, E_2, \ldots, E_k\}$$

For example, could be given $5$ maps:

$$\mathcal{E} = \{E_1, E_2, E_3, E_4, E_5\}$$

$X$ is all $(q, E_i)$ in which $(q_x, q_y) \in E_i$ and $E_i \in \mathcal{E}$.

Recall the common structure.

Given an infinite *map family*, $\mathcal{E}$, of environments.

Examples:

- The set of all connected, bounded polygonal subsets that have no interior holes (formally, they are *simply connected*).
- The previous set expanded to include all cases in which the polygonal region has a finite number of polygonal holes.
- All subsets of $\mathbb{R}^2$ that have a finite number of points removed.
- All subsets of $\mathbb{R}^2$ that can be obtained by removing a finite collection of nonoverlapping discs.
- All subsets of $\mathbb{R}^2$ obtained by removing a finite collection of nonoverlapping convex sets.
- A collection of piecewise-analytic subsets of $\mathbb{R}^2$.

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# State Space For Unknown Map



In spite of larger $\mathcal{E}$, there is no difference:

$X$ is all pairs $(q, E)$ in which $(q_x, q_y) \in E$ and $E \in \mathcal{E}$.

We can write $X \subset \mathbb{R}^2 \times S^1 \times \mathcal{E}$.

$X$ is enormous! But that is fine here. We do not compute directly on it.

Note: Putting useful probability densities over $X$ might be difficult or impossible.

$X$ is usually **not a manifold** (doesn't look like C-space)

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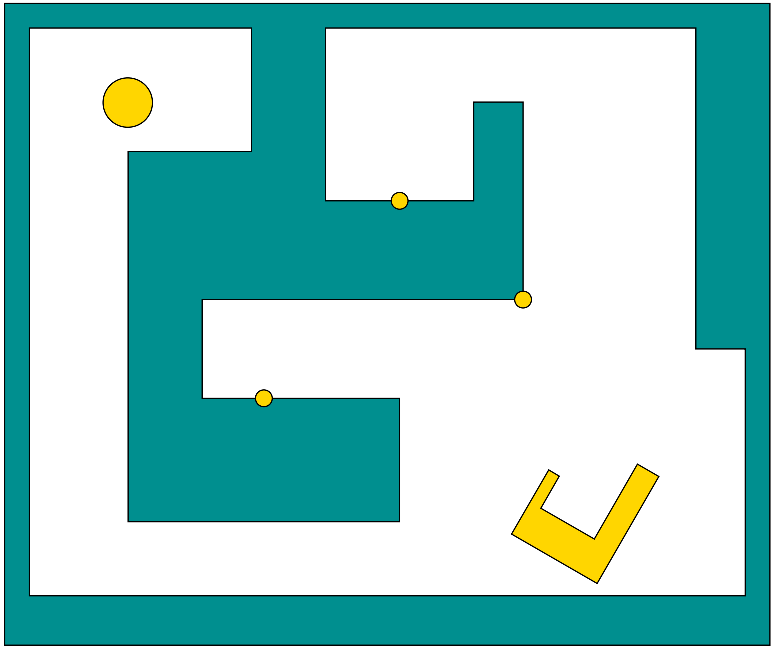
Place a *body* $B$ into $E$.



Each could have a configuration space $SE(2)$, so that we transform it: $B(q_x, q_y, q_\theta) \subset E$.

Here, assume every body is a point, except for obstacles.
Otherwise, see Chapter 4 of *Planning Algorithms* for configuration space obstacles.

# Terms for Bodies



- **Robot:** A body that carries sensors, performs computations, and executes motion commands.
- **Landmark:** Usually a small body that has a known location and is easily detectable and distinguishable from others.
- **Object:** A body that can be detected and manipulated by a robot. It can *carried* by a robot or *dropped* at a location.
- **Pebble:** A small object that is used as a marker to detect when a place has been revisited.
- **Target:** A person, a robot, or any other moving body that we would like to monitor using a sensor.
- **Obstacle:** A fixed or moving body that obstructs the motions of others.
- **Evader:** An unpredictable moving body that attempts to elude detection.
- **Treasure:** Usually a stationary body that has an unknown location but is easy to recognize by a sensor directly over it.
- **Tower:** A body that transmits a signal, such as a cell-phone tower or a lighthouse.

# Important Body Properties



Names of bodies are not important.

Instead, the properties that affect mathematical models are crucial:

1. What are its *motion capabilities*?
2. Can it be *distinguished* from other bodies?
3. How does it *interact* with other bodies?

Motion

Distinguishability

Interaction

# Body Property 1: Motion Capabilities



There are three possibilities:

1. If **static**, then it never moves.
   *Examples: Most landmarks, obstacles, a tower*

2. It may have **predictable** motion.
   *Examples: A rolling ball, a pendulum, a robot*

3. It may have **unpredictable** motion.
   *Examples: An evader, a target*

If planning is involved, then another issue is whether or not the body can be commanded to move.

Take any collection of distinct bodies $B_1, \ldots, B_n$.

Let $\sim$ be any equivalence relation:
$B_i \sim B_j$ *if and only if they cannot be distinguished from each other.*

Example: Could assign *labels* to be bodies. With humans, we have *women* and *men*.

Warning: Sometimes indistinguishability might not be transitive!

Three interaction types are generally possible between a pair $B_1$, $B_2$, of bodies:

- **Sensor obstruction:** Suppose a sensor would like to observe information about body $B_1$. Does body $B_2$ interfere with the observation?
- **Motion obstruction:** Does body $B_2$ obstruct the possible motions of body $B_1$? If so, then $B_2$ becomes an obstacle that must be avoided.
- **Manipulation:** In this case, body $B_1$ could cause body $B_2$ to move. For example, if $B_2$ is an obstacle, then $B_1$ might push it out of the way.

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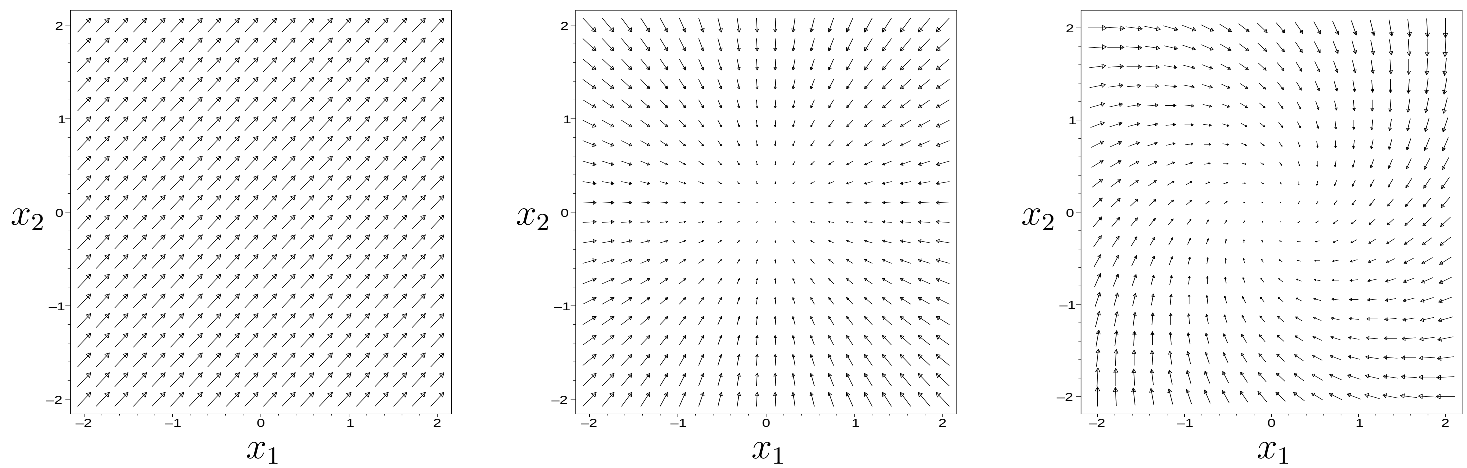
# Fields



A *field* is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, with $n = 2$ or $n = 3$ and $m \leq n$.




Examples:

- Encoding $E$: $f : \mathbb{R}^2 \rightarrow \{0, 1\}$ in which $f(q_x, q_y) = 1$ if and only if $(q_x, q_y) \in E$.
- An altitude map: $f : \mathbb{R}^2 \rightarrow [0, \infty)$.
- An intensity field: $f : \mathbb{R}^2 \rightarrow [0, \infty)$.
- An electromagnetic field: $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

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Of course the world is not static.

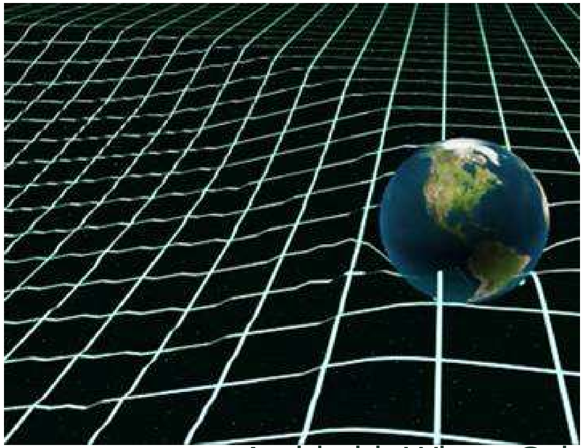
Let $T$ be *time interval*; usually, $T = [0, \infty)$.

Using any state space $X$, define *state-time space*:

$$Z = X \times T$$

Each $z \in Z$ is a pair $z = (x, t)$ and $x$ is the state at time $t$

No, not this:

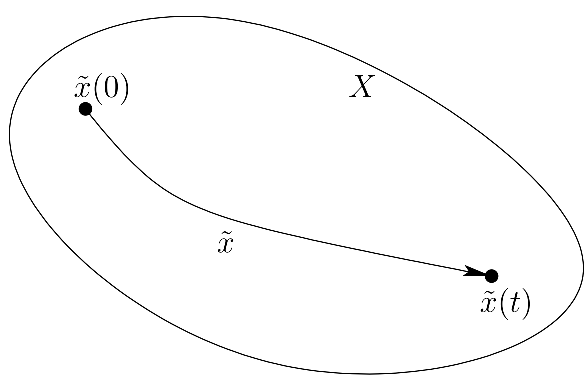


# State Trajectories

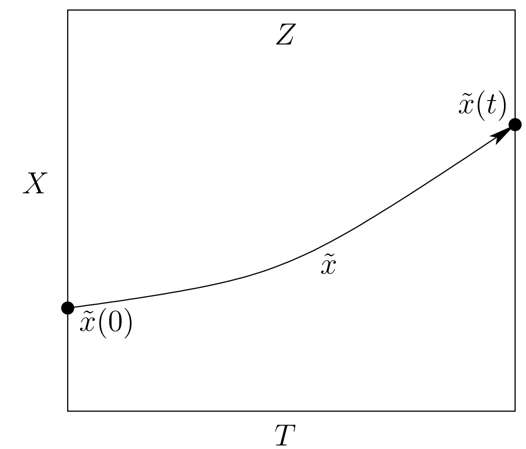
A *state trajectory* $\tilde{x}$ is a time-parameterized path through $X$;

$$\tilde{x} : T \to X$$

Sometimes, domain of $\tilde{x}$ may be only $[0, t]$.




Mapping into $X$




Mapping into $Z$

Could take time derivatives of states and expand state space. We will not do that here.

# Sensor mapping

Let $X$ be any physical state space.

Let $Y$ denote the *observation space*, which is the set of all possible sensor observations.

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A virtual sensor is defined by a *sensor mapping*:

$$h : X \to Y.$$

Note similarity to transfer function for physical sensors.

When $x \in X$, the sensor instantaneously observes $y = h(x) \in Y$.

Using the sensor mapping, we will make many models:

- Basic (boring) examples
- Depth sensors
- Detection sensors
- Relational sensors
- Gap sensors
- Field sensors

Purpose: To define models of *information* to be used in filters.

Remember: Virtual sensors could have many physical implementations.

# Basic Examples

**The weakest possible sensor**

DUMMY SENSOR:
$Y = \{0\}$ and $h(x) = 0$ for all $x \in X$

---

**The strongest possible sensor(s)**

IDENTITY SENSOR:
$Y = X$ and $y = h(x) = x$
Just give me the state!

BIJECTIVE SENSOR:
$h$ is bijective function from $X$ to $Y$.
$x$ can be reconstructed as $x = h^{-1}(y)$.

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$X = Y = \mathbb{R}^3$

LINEAR SENSOR:
Let $y = h(x) = Cx$ for $3$ by $3$ matrix $C$.

If $C$ has full rank, then $h$ is a bijective sensor.
If $C$ has lower rank, then lines or planes produce same observation.

Linear sensors used widely in control theory.

PROJECTION SENSOR:

Choose some components of $X$.

---

$X = \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in X$.

$Y = \mathbb{R}^2$

$y = h(x) = (x_1, x_2)$

---

$X = \mathbb{R}^2 \times S^1$

A state is $(q_x, q_y, q_\theta) \in X$.

Position sensor: Observes $(q_x, q_y)$ and leaves $q_\theta$ unknown.

Ideal compass: Observes $q_\theta$ and leaves $q_x$ and $q_y$ unknown.

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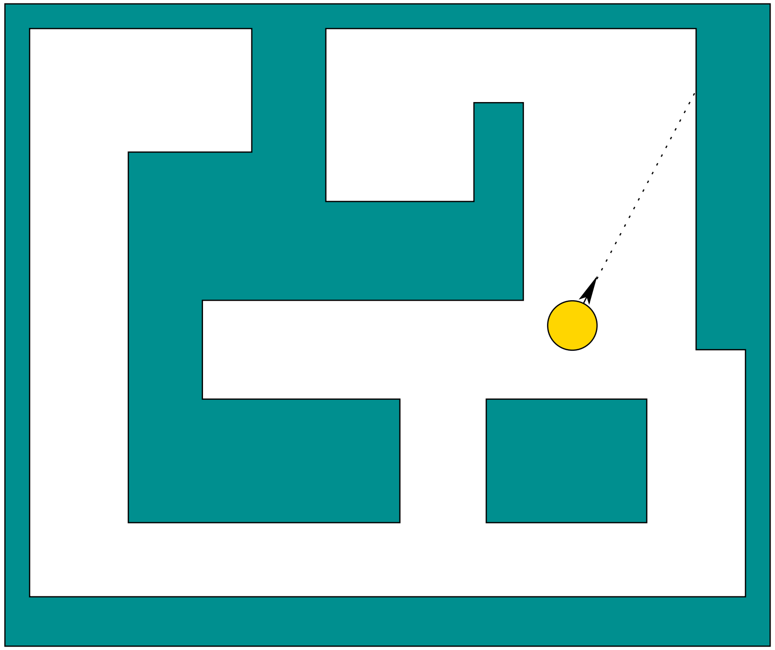
# Depth sensors

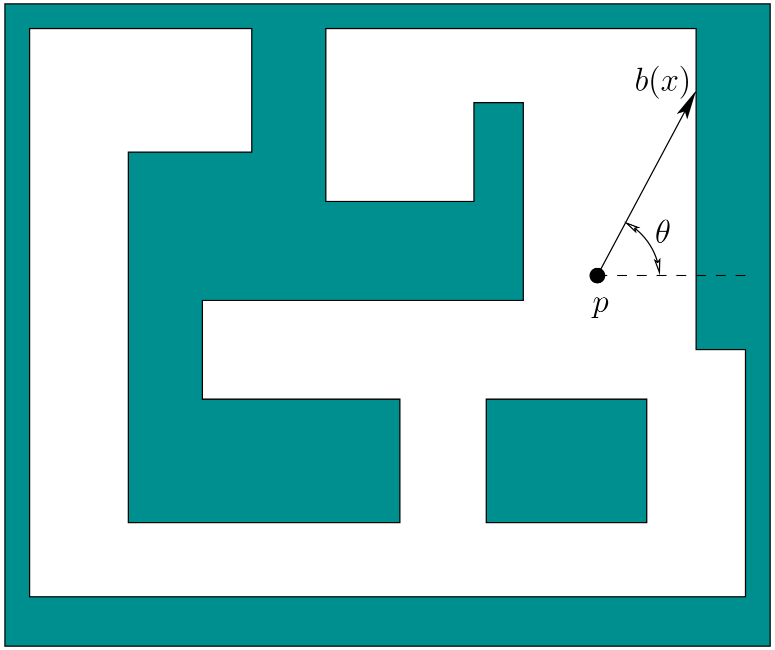


Observe the distance to the boundary of $E$.

State space: $X \subset SE(2) \times \mathcal{E}$
State: $x = (q_x, q_y, q_\theta, E)$ with $(q_x, q_y) \in E$ and $E \in \mathcal{E}$.

# Directional Depth Sensor





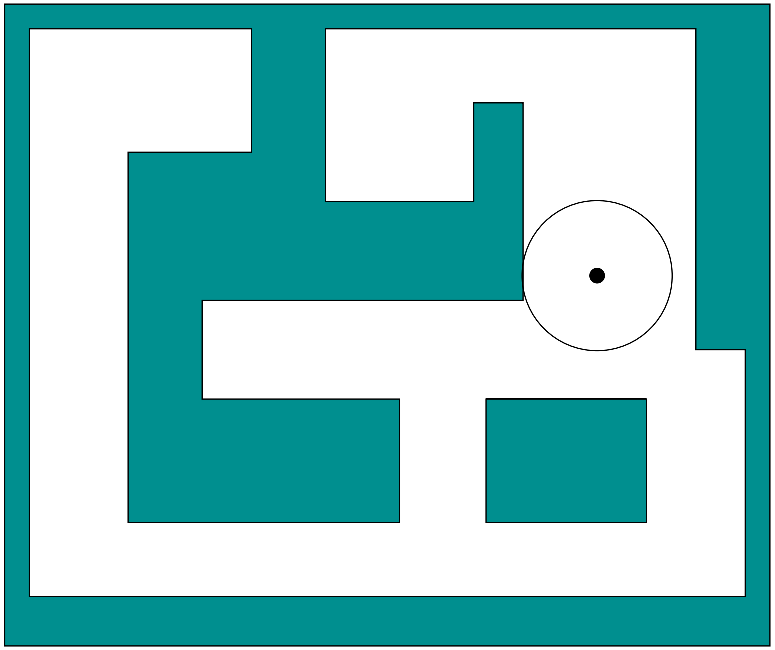

DIRECTIONAL DEPTH SENSOR:

$$h_d(p, \theta, E) = \|p - b(x)\|$$

Let $p = (q_x, q_y)$ and $\theta = q_\theta$ (shorthand notation)
$b(x)$ is point on boundary $\partial E$ hit by ray.

# Boundary Distance Sensor





BOUNDARY DISTANCE SENSOR:

$$h_{bd}(p, \theta, E) = \min_{\theta' \in [0, 2\pi)} h_d(p, \theta', E)$$

No dependency on $\theta$

# Proximity and Boundary Sensors



Fix some $\epsilon > 0$.

PROXIMITY SENSOR:

$$h_{p\epsilon}(p, \theta, E) = \begin{cases} 1 & \text{if } h_{bd}(p, \theta, E) \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Detects whether within $\epsilon$ of the boundary.

---

BOUNDARY SENSOR:

$$h_{bd}(p, \theta, E) = \begin{cases} 1 & \text{if } h_{bd}(p, \theta, E) = 0 \\ 0 & \text{otherwise} \end{cases}$$

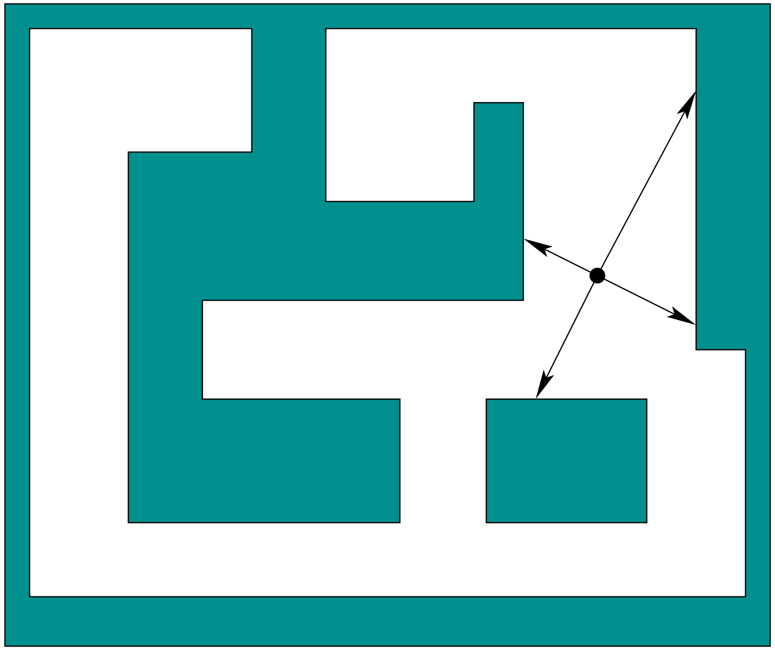
Detects whether boundary is contacted.

SHIFTED DIRECTIONAL DEPTH SENSOR:

Robot oriented along $\theta$, but sensor is offset by $\phi$

$$h_{sd\phi}(p, \theta, E) = \|p - b(p, \theta + \phi,\ E)\|$$

# K-Directional Depth Sensor



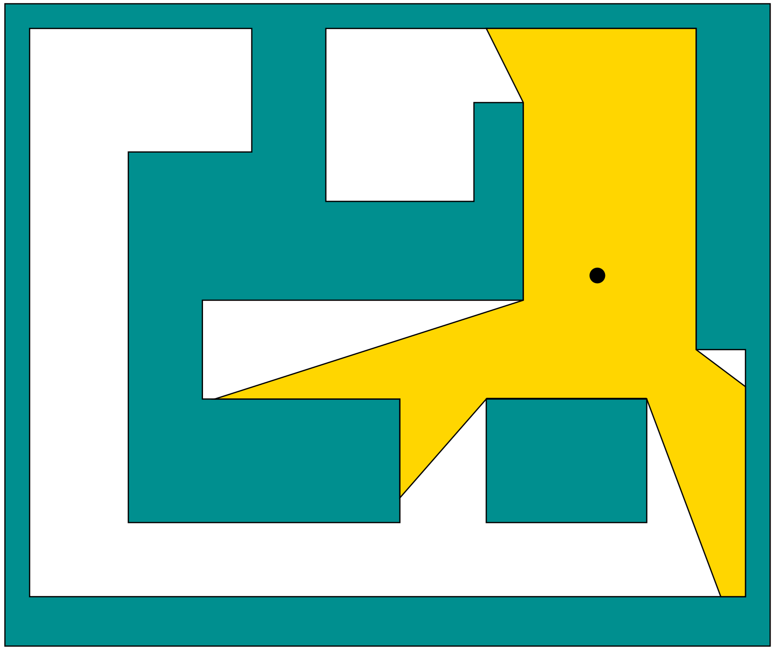
K-DIRECTIONAL DEPTH SENSOR:

Let $k$ be number of directions.

The observation is a vector $y = (y_1, \ldots, y_k)$

$$y_i = h_i(p, \theta, E) = h_{sd\phi_i}(p, \theta, E).$$

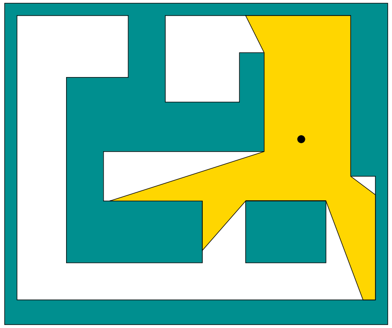
Like an infinite-dimensional vector of observations



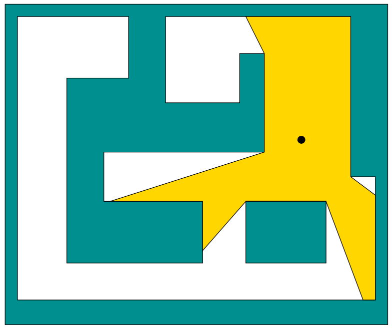
OMNIDIRECTIONAL DEPTH SENSOR:

$h_{od}(x) = y$, in which $y : S^1 \to [0, \infty)$

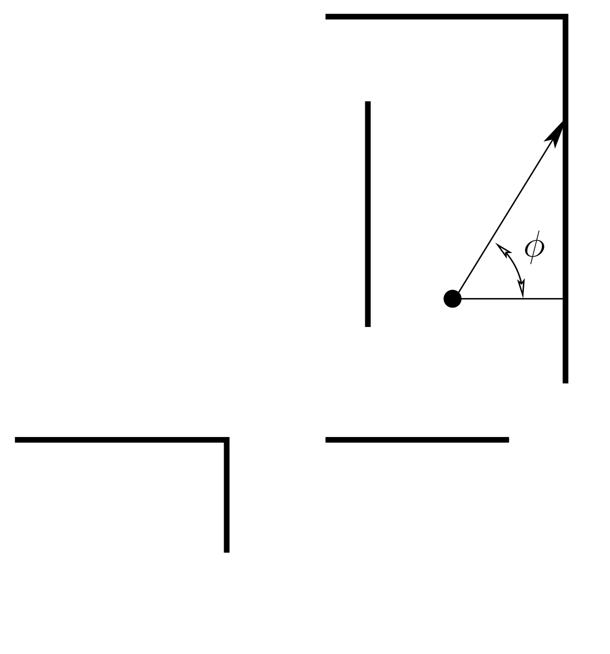
$$y(\phi) = h_{od\phi}(p, \theta, E).$$



How does the observation $y : S^1 \rightarrow [0, \infty)$ look?



How does the observation $y : S^1 \rightarrow [0, \infty)$ look?



Limited angle:

$$y : [\phi_{min}, \phi_{max}] \rightarrow [0, \infty)$$

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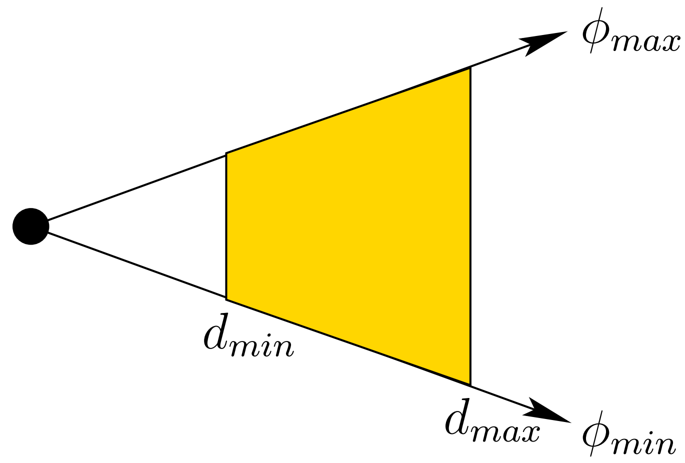
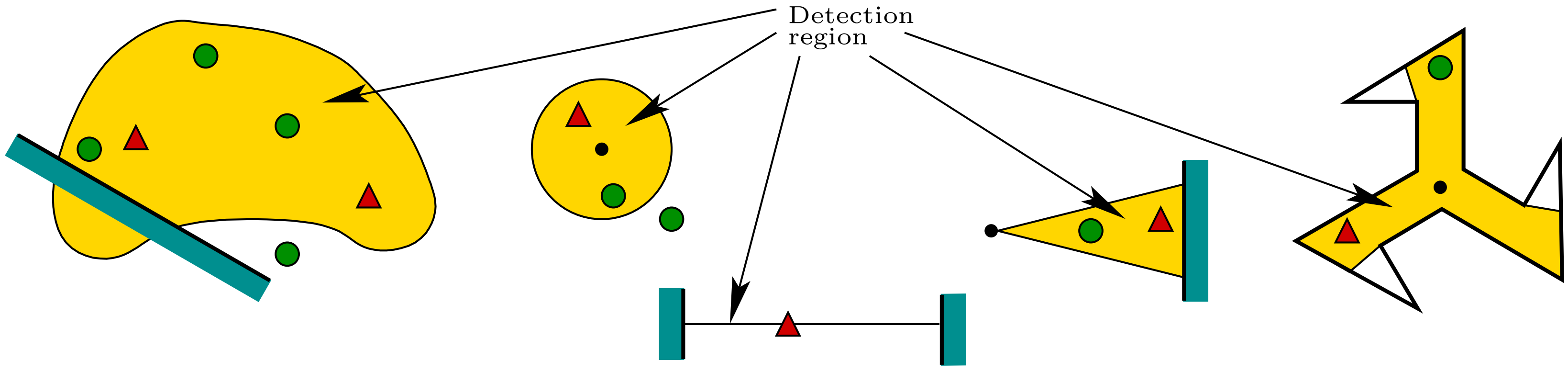
Limited depth:

$$h_{dd}(p, \theta, E) = \begin{cases} d(x) & \text{if } d_{min} \leq d(x) \leq d_{max} \\ \# & \text{otherwise} \end{cases}$$



# Detection sensors

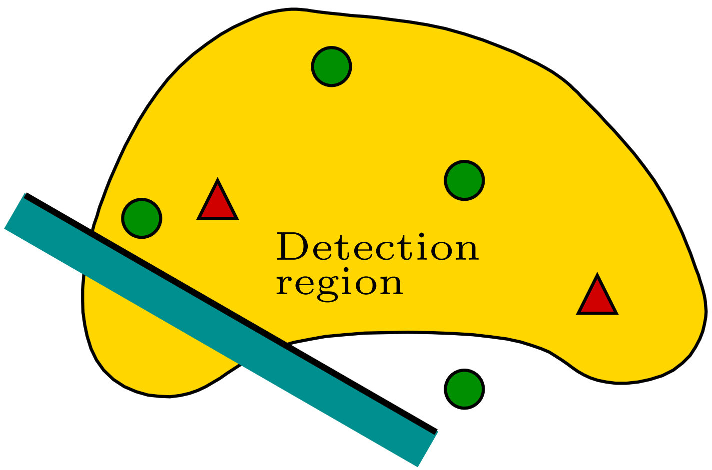
New category: Detection Sensors




Is a body in the field of view, or *detection region*?

# Detection Sensors: Fundamental Aspects






Three fundamental questions:

1. Can the sensor move? For example, it could be mounted on a robot or it could be fixed to a wall.
2. Are the bodies so large relative to the range of the sensor that the body models cannot be simplified to points?
3. Can the sensor provide additional information that helps to classify a body within its detection region?

Simplest case: Answer "no" to all three questions.

# Static Binary Detector

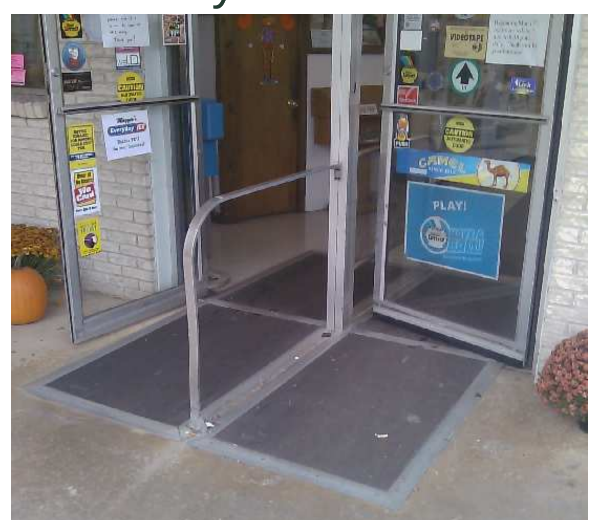


This is the simplest case.

STATIC BINARY DETECTOR:

$$h(p, E) = \begin{cases} 1 & \text{if } p \in V \\ 0 & \text{otherwise} \end{cases}$$

Simply indicates whether the body is in $V$



# Moving Binary Detector



$q$ is configuration of the body carrying the sensor.

$V(q)$ is the configuration-dependent detection region.

MOVING BINARY DETECTOR:

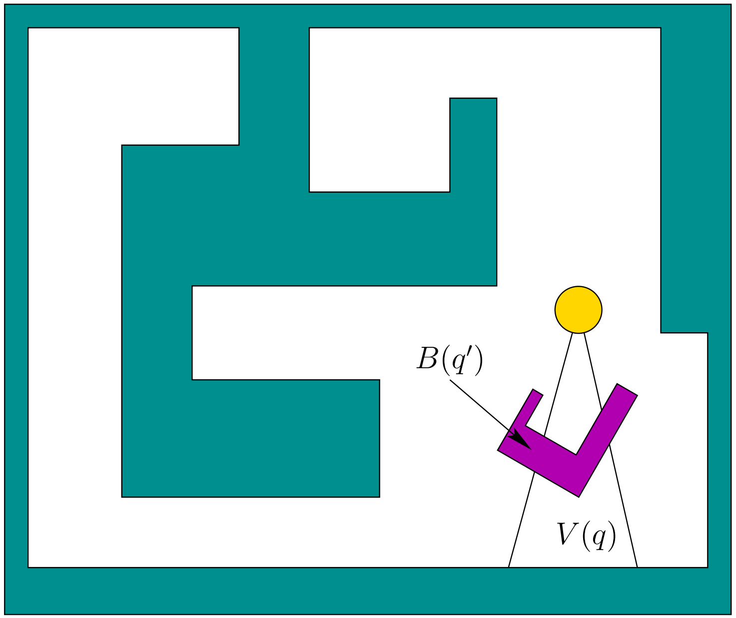
$$h(p, E) = \begin{cases} 1 & \text{if } p \in V(q) \\ 0 & \text{otherwise} \end{cases}$$

$V$ has simply been replaced by $V(q)$

A body has configuration $q'$ and $B(q') \subset E$.

DETECTING LARGER BODIES:

$$h(q', E) = \begin{cases} 1 & \text{if } B(q') \cap V \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$




Looks like obstacle regions in configuration space!

There are $n$ points bodies, $P = \{p_1, \ldots, p_n\}$.

State: $x = (q, p_1, \ldots, p_n, E)$, in which $q$ is sensor configuration.

AT-LEAST-ONE-BODY DETECTOR:

$$h(q, p_1, \ldots, p_n, E) = \begin{cases} 1 & \text{if for any } i,\ p_i \in V(q) \\ 0 & \text{otherwise} \end{cases}$$

Sensor detects when at least one of the bodies is in $V(q)$.

BODY COUNTER:

$$h(q, p_1, \ldots, p_n, E) = |P \cap V(q)|$$

Shopping mall

Coral reef

If number of bodies generally unknown, but sensors fixed and environment $E$ known:

$$X = \{\#\} \cup E \cup E^2 \cup E^3 \cup E^4 \cdots$$

$L$ is a set of class labels.

$\ell$ is an assignment mapping:

$$\ell : \{1, \ldots, n\} \rightarrow L$$

LABELED-BODY DETECTOR:

$$h_\lambda(p, E) = \begin{cases} 1 & \text{if for some } i,\ p_i \in V \text{ and } \ell(i) = \lambda \\ 0 & \text{otherwise} \end{cases}$$

Examples: Each body is a man, dog, tree, car, ...

# Relational sensors

# Relational Sensors



Consider any relation $R$ on the set of all bodies.

For a pair of bodies, $B_1$ and $B_2$, examples of $R(B_1, B_2)$ are:

- $B_1$ is in front of $B_2$
- $B_1$ is to the left of $B_2$
- $B_1$ is on top of $B_2$
- $B_1$ is closer than $B_2$
- $B_1$ is bigger than $B_2$.

More precisely, Let $R_x(i, j)$ mean $B_i$ is related to $B_j$, when the system is at state $x$.

Idea is due to Guibas

# Primitive Relational Sensor

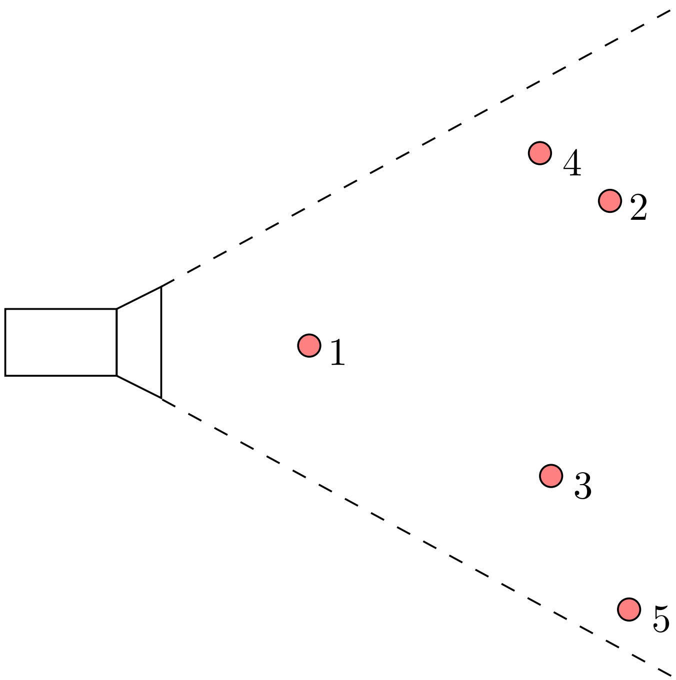
PRIMITIVE RELATIONAL SENSOR:

$$h(x) = \begin{cases} 1 & \text{if } R_x(i, j) \\ 0 & \text{otherwise} \end{cases}$$

Simply detects whether the relation is satisfied for bodies $B_i$ and $B_j$.

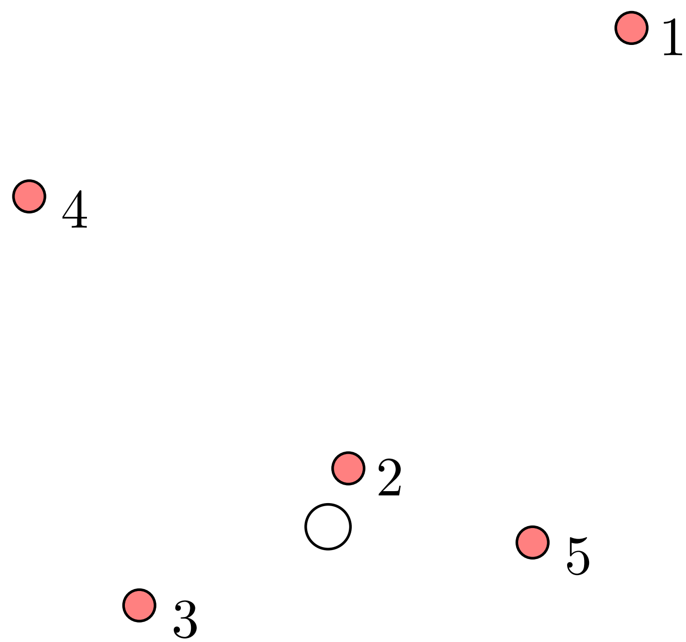
Using this, we can form *compound relational sensors*.

Relation: “is to the left of”




Observation: $y = (4, 2, 1, 3, 5)$
Observation space: $Y$ is all $5!$ permutations.

Relation: “is closer than”

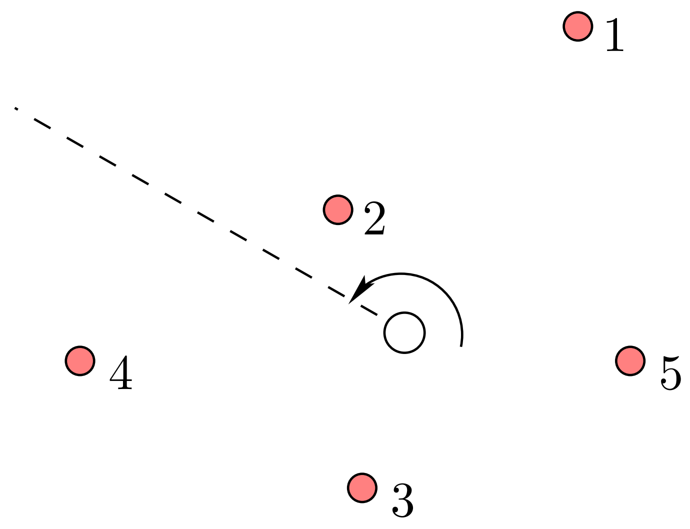



Observation: $y = (2, 3, 5, 4, 1)$

# Cyclic Permutation Sensor



Relation: "is to the left of in counterclockwise order"




Observation: $y = (1, 2, 4, 3, 5)$

Note that $y$ could equivalently be $(4, 3, 5, 1, 2)$.

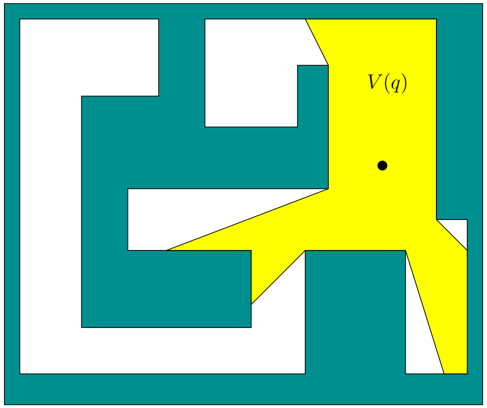
# Gap sensors

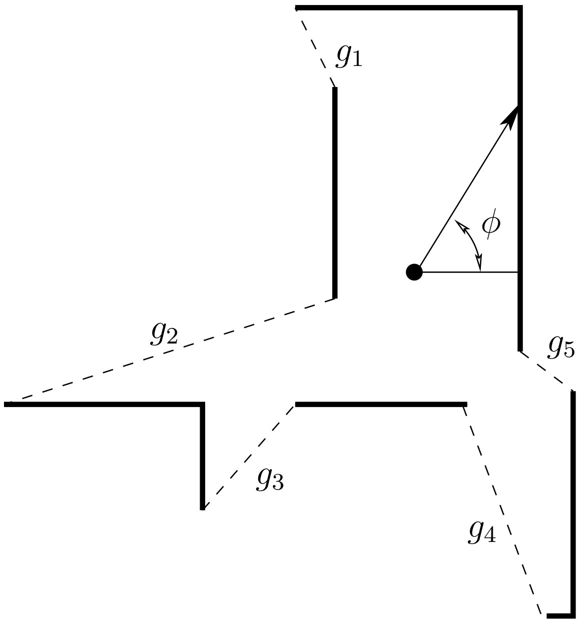
Report information obtained along the boundary of $V(q)$, which is denoted as $\partial V(q)$

Two qualitatively different parts of $\partial V(q)$:

1. A piece of a body boundary
2. A gap (discontinuity in depth)

A gap sensor reports how these parts alternate.



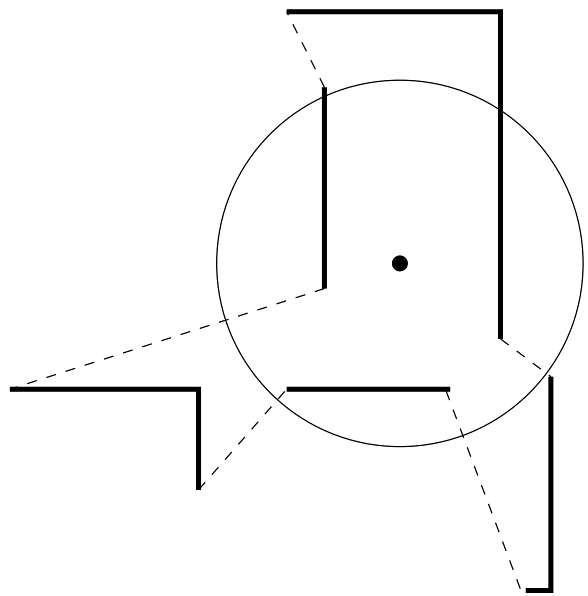

SIMPLE GAP SENSOR:

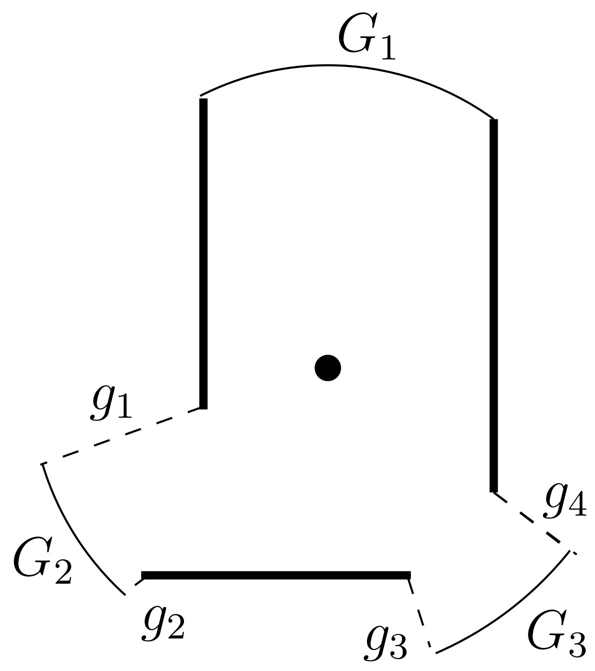
Alternating between boundary and gaps:

$$y = (B_0, g_1, B_0, g_2, B_0, g_3, B_0, g_4, B_0, g_5)$$

Equivalently:

$$y = (g_1, g_2, g_3, g_4, g_5)$$



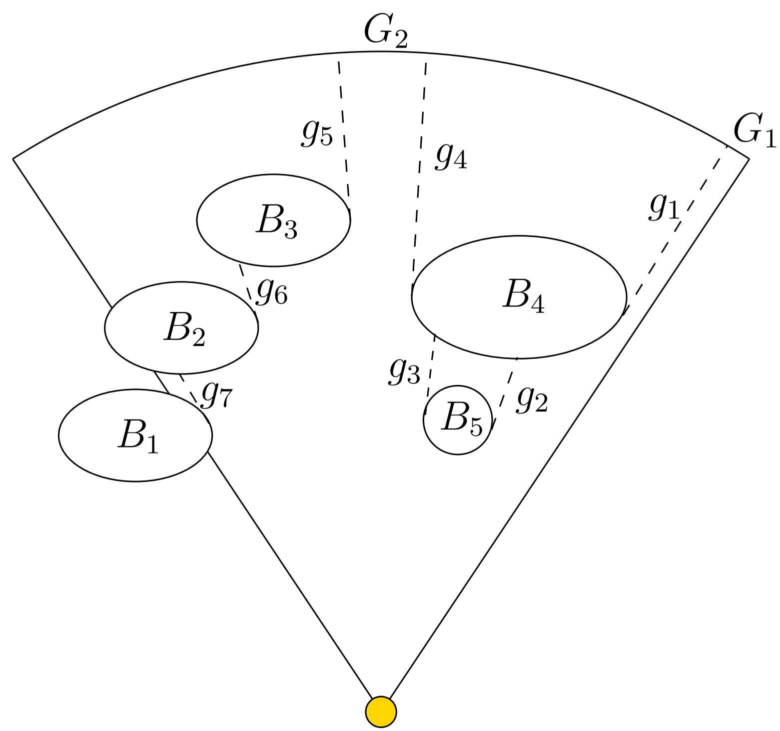



A new kind of gap, due to being out of range: $G_i$

DEPTH-LIMITED GAP SENSOR:

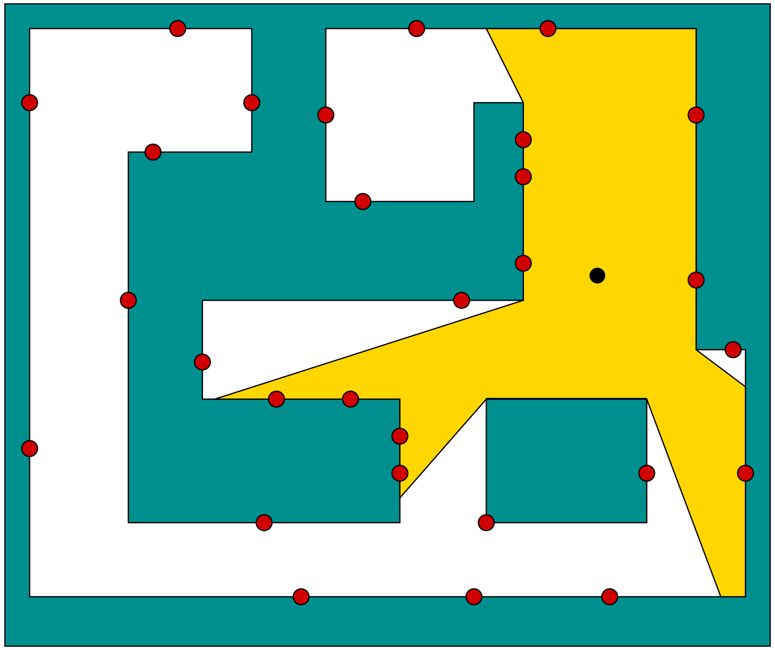
$y = (B_0, G_1, B_0, g_1, G_2, g_2, B_0, g_3, G_3, g_4)$

# Multibody Gap Sensor




MULIBODY GAP SENSOR:

$$y = (G_1, g_1, B_4, g_2, B_5, g_3, B_4, g_4, G_2, g_5, B_3, g_6, B_2, g_7, B_1)$$

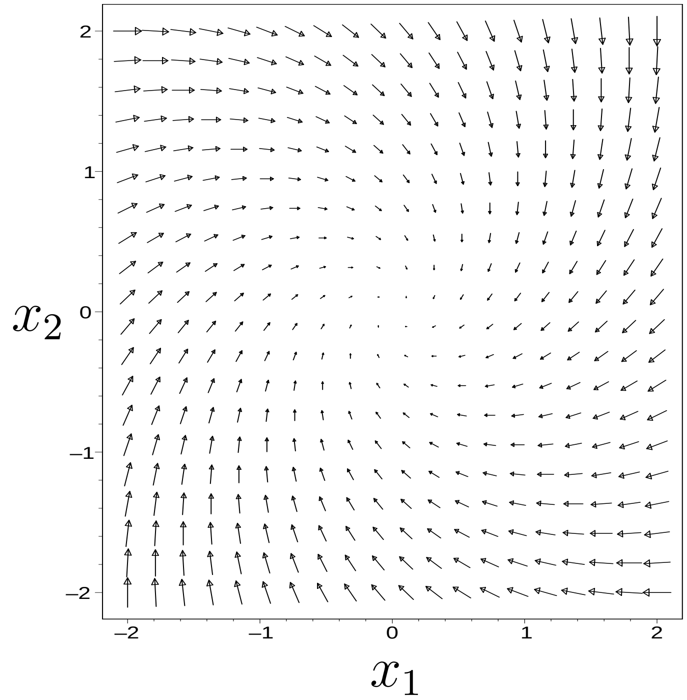


LANDMARK COUNTER:

$y = (3, 3, 4, 0, 1)$

Equivalent to combinatorial visibility vector from Gfeller et al. 2007.

# Field sensors




DIRECT FIELD SENSOR:

$$h(x) = h(p, \theta) = (f_1(p), f_2(p))$$

Vectors appear with respect to global frame orientation.

## DIRECT INTENSITY SENSOR:

$$h(x) = h(p, \theta) = \|f(p)\|$$

## INTENSITY ALARM:

$$h(p, \theta) = \begin{cases} 1 & \text{if } \|f(p)\| \geq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

Unknown monotonically increasing function:

$$g : [0, \infty) \to [0, \infty)$$

TRANSFORMED INTENSITY:

$$h(x) = g(\|f(x)\|)$$

# Field Vector Observation



More realistically, vectors observed in local orientation frame.

$R(\phi)$ is $2 \times 2$ rotation matrix by $\phi$.

FIELD VECTOR OBSERVATION:

$$h_{fv}(x) = R(-\theta) f(p)$$

If $f$ is given and $\theta$ is unknown, then it can be determined using $h_{fv}(x)$.

Likewise, if $\theta$ is known and $f$ is unknown, then $f(p)$ can be determined from $f(p) = R(\theta) h_{fv}(x)$.

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# Field Direction Observation



Let $y' = h_{fv}(x)$.

FIELD DIRECTION OBSERVATION:

$$y = h_{fdo}(x) = \text{atan2}(y'_2, y'_1)$$

Special case: An ideal magnetic compass, $f(p) = (0, 1)$.

The orientation $\theta$ can be recovered from the given field.

# Preimages

The amount of state uncertainty due to a sensor

$$h : X \to Y$$

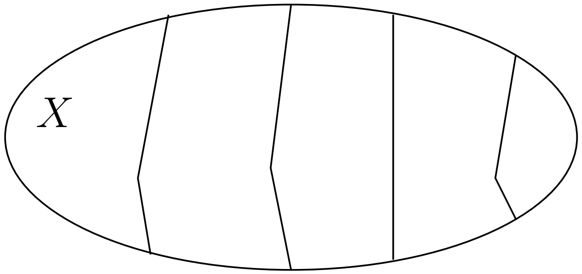
The preimage of an observation $y$ is

$$h^{-1}(y) = \{x \in X \mid y = h(x)\}$$

Think about the uncertainty being handled here!

Suppose $X$ and $h : X \rightarrow Y$ are given.

The set of all preimages partitions $X$



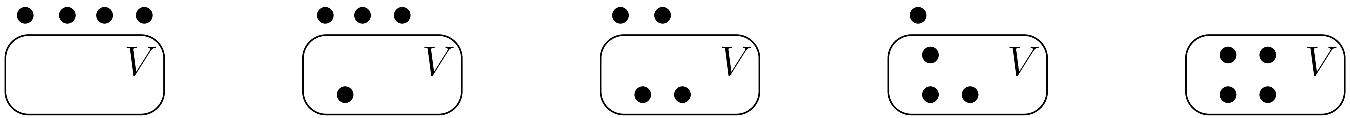

There is one preimage for every $y \in Y$.

Let $\Pi(h)$ be the partition $X$ that is induced by $h$.

# Detection Sensor Example of $\Pi(h)$



- $n$ point bodies move in $\mathbb{R}^2$.
- $X = \mathbb{R}^{2n}$
- $Y = \{0, 1, \ldots, n\}$
- The sensor mapping $h : X \rightarrow Y$ counts how many points lie a fixed detection region $V$.



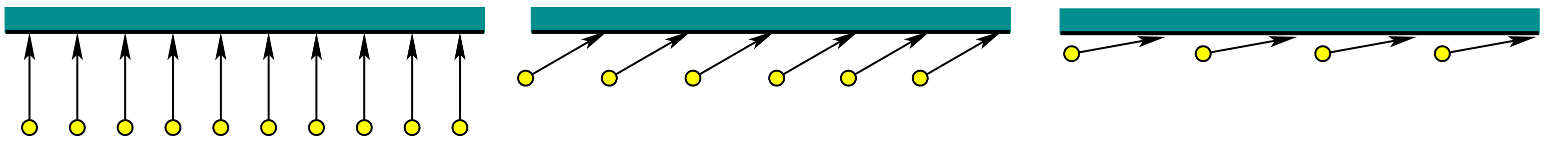

For $n = 4$, there are $5$ equivalence classes in $\Pi(h)$.

# Depth Sensor Example of $\Pi(h)$



Recall directional depth sensor.

For a known environment, $X = E \times S^1$.



The preimages are chunks of $SE(2)$.

---

What happens for an unknown environment?

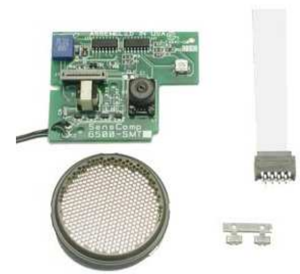
The preimages are chunks of $\mathbb{R}^2 \times S^1 \times \mathcal{E}$.

# Sensor lattice

# Comparing the Power of Sensors



Question: Is 

better than  ?

It is better to compare *virtual* sensors...

---

Fix the state space $X$.

Take any two sensors, $h_1 : X \rightarrow Y_1$ and $h_2 : X \rightarrow Y_2$.

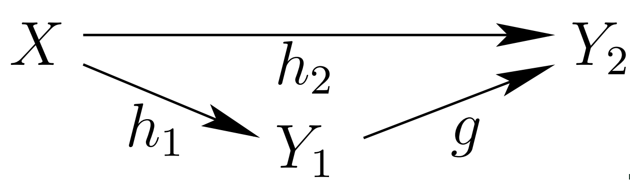
$h_1$ *dominates* $h_2$ if and only if $\Pi(h_1)$ is a refinement of $\Pi(h_2)$.
This is denoted as $h_1 \succeq h_2$.

# Comparing the Power of Sensors

If $h_1 \succeq h_2$, then $h_2$ can be "simulated" using only observations from $h_1$.

If $\Pi(h_1)$ is a refinement of $\Pi(h_2)$, then we can figure out what observation $h_2$ must make, using only $y_1$.

This is interpreted as the existence of a function $g : Y_1 \to Y_2$.




What about computability or complexity of $g$?

# Comparing More Sensors



Fix the state space $X$

We could have a *sensor chain:*

$$h_1 \succeq h_2 \succeq h_3 \succeq h_4 \succeq h_5$$

# Comparing More Sensors



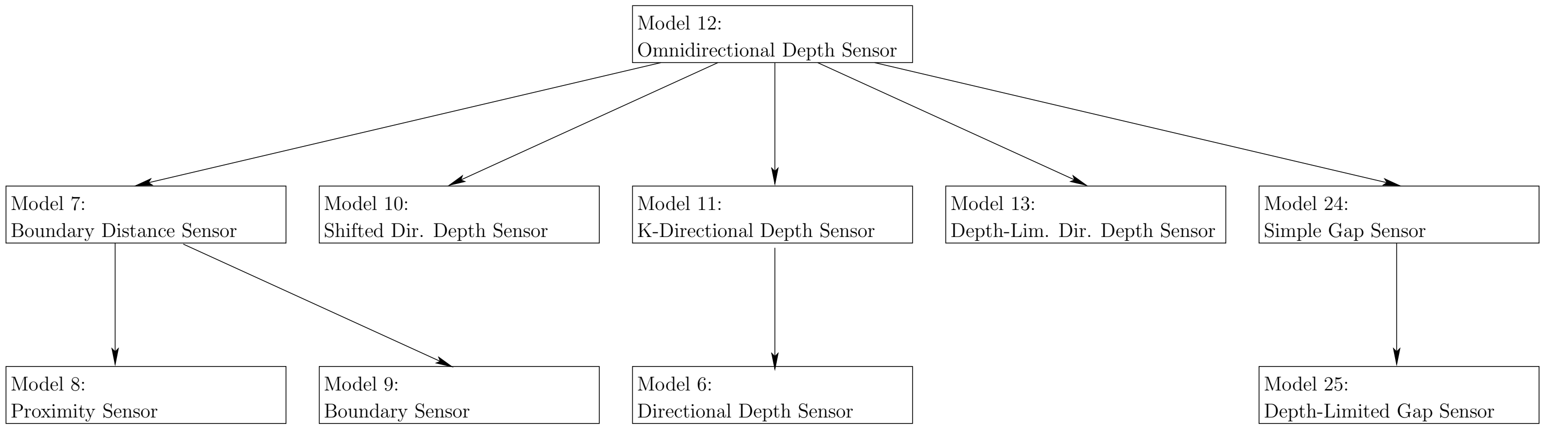
Fix the state space $X$

We could have a *sensor chain:*

$$h_1 \succeq h_2 \succeq h_3 \succeq h_4 \succeq h_5$$

---

We could have a *sensor tree:*

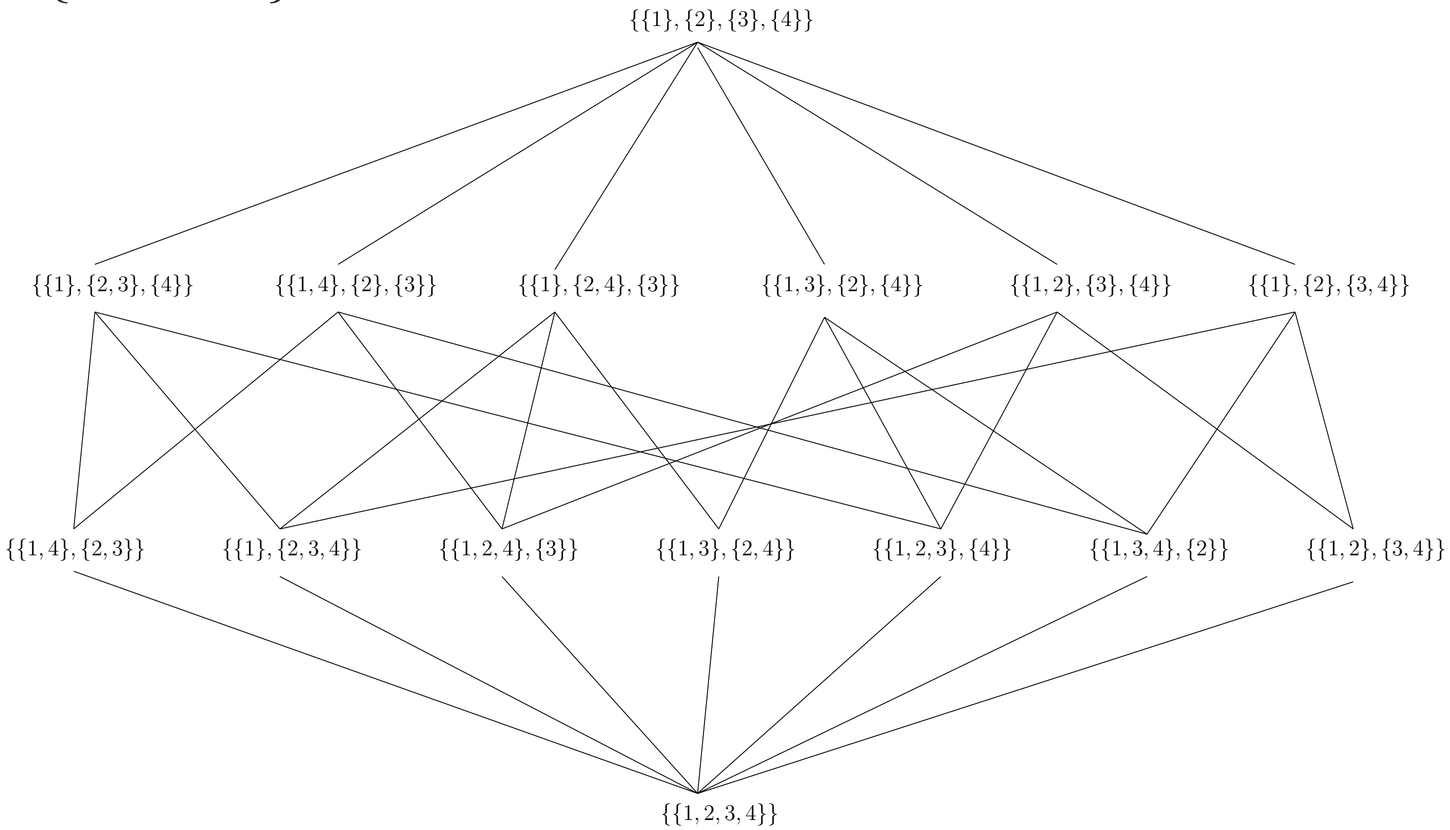



Could we even have a *directed acyclic graph*?

# A Lattice of Partitions

For any set $X$, the set of all partitions forms a complete lattice.

$$X = \{1, 2, 3, 4\}$$




Every pair has a glb and lub.

# The Sensor Lattice



Fix $X$ and consider the set of *all* possible sensors $h : X \rightarrow Y$.

Above, $Y$ is not fixed!!

---

We say two sensors $h_1$ and $h_2$ are *equivalent* if and only if $\Pi(h_1) = \Pi(h_2)$.

Really, the partition of $X$ *is* the sensor model.

---

The set of all partitions of $X$ forms the *sensor lattice*.

All sensor models embed into this lattice!

The bijective sensor and dummy sensor are at the top and bottom, respectively.

# Additional complications

# Nondeterministic Disturbance



Nondeterministic sensor mapping:

$$h : X \to \mathrm{pow}(Y)$$

---

Corresponding preimage definition:

$$h^{-1}(y) = \{x \in X \mid y \in h(x)\}$$

A sensor mapping induces a cover $\mathcal{C}(h)$ of $X$, instead of a partition.

ONE-DIMENSIONAL POSITION SENSOR:

$$h(x) = \{y \in Y \mid |x - y| \leq \epsilon\}$$

For example, $h(2) = [2 - \epsilon, 2 + \epsilon]$

The preimage of an observation $y$ is

$$h^{-1}(y) = \{x \in X \mid |x - y| \leq \epsilon\}.$$

Clearly, a cover of $X = \mathbb{R}$ is induced by $h$.

# Faulty Binary Detection Sensor



Two kinds of mistakes:

1. **False positive:** $h(p, E) = 1$ even though $p \notin V$
2. **False negative:** $h(p, E) = 0$ even though $p \in V$

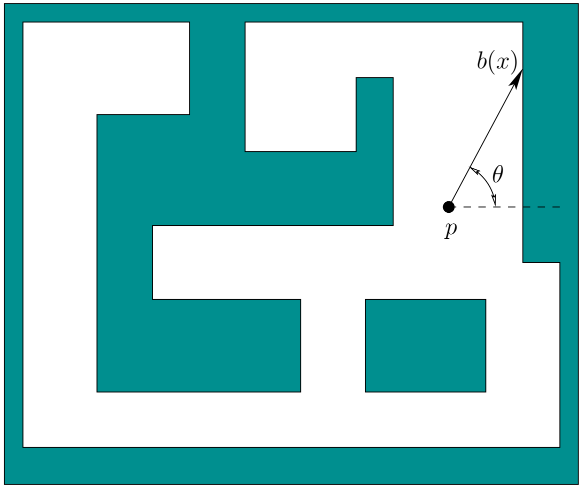
What does $\mathcal{C}(h)$ look like?

$X$ is completely covered by two preimages.

Fix accuracy, $\epsilon \geq 0$.

INACCURATE DIRECTIONAL DEPTH SENSOR:

$$h_\epsilon(p, \theta, E) = \{y \in [0, \infty) \mid |||p - b(x) - y||| \leq \epsilon\}.$$



# Probabilistic Disturbances



The sensor mapping is replaced by:

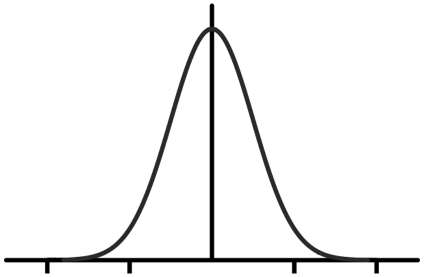
$$p(y|x)$$

Using nondeterministic $h$ that we can declare $p(y|x) = 0$ for all $y \notin h(x)$.

Error model: Gaussian with zero mean and variance $\sigma^2$

PROBABILISTIC 1D POSITION SENSOR:

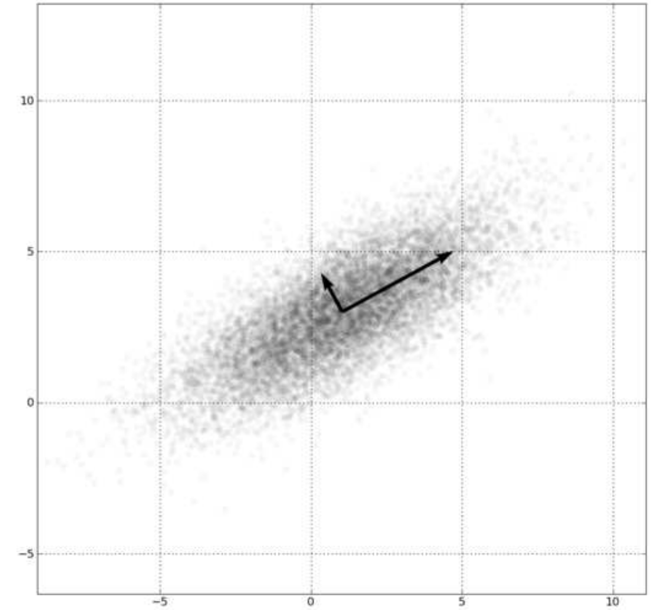
$$p(y|x) = \frac{1}{|\Sigma|^{1/2}(2\pi)^{k/2}} e^{(y-x)^T \Sigma^{-1} (y-x)}.$$



Error model: Gaussian with zero mean and $\Sigma$ as a $k \times k$ covariance matrix.

PROBABILISTIC GENERAL POSITION SENSOR:

$$p(y|x) = \frac{1}{|\Sigma|^{1/2}(2\pi)^{k/2}} e^{(y-x)^T \Sigma^{-1} (y-x)}.$$



# Probabilistic Detector



Now probabilities are assigned to false positives and false negatives.

False positive probability:
$p(y = 1 \mid p \notin V)$

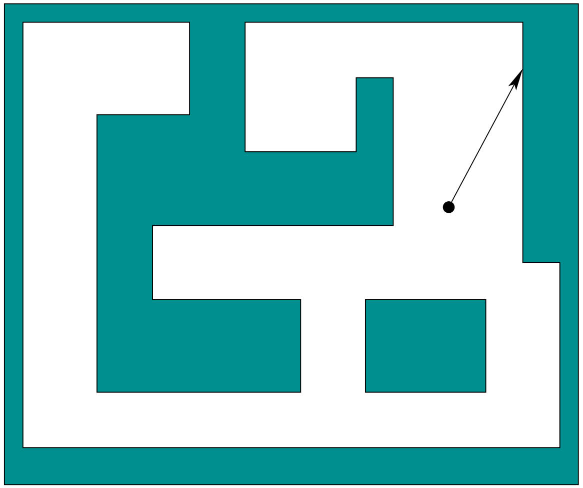
False negative probability:
$p(y = 0 \mid p \in V)$

If these probabilities are small, then the sensor is quite informative.

Again assume zero-mean Gaussian error density.

PROBABILISTIC DIRECTIONAL DEPTH SENSOR:

$$p(y|p, \theta, E) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\|p-b(x)\|)^2}{2\sigma^2}}$$



Recall state-time space, $Z = X \times T$.

Sensor mapping:

$$h : Z \to Y$$

$y = h(z)$, or equivalently, $y = h(x, t)$

Consider preimages, partitions of $Z$, and sensor lattice.

$$h^{-1}(y) = \{(x, t) \in Z \mid y = h(x, t)\}$$

PERFECT CLOCK MODEL:

$$y = h(z) = h(x,t) = t.$$

## DETECTOR WITH TIME STAMP:

$$h(p, E, t) = \begin{cases} (1, t) & \text{if } p \in V \text{ at time } t \\ (0, t) & \text{otherwise} \end{cases}$$

# History-Based Sensors



State trajectory: $\tilde{x} : [0, t] \rightarrow X$
Let $\tilde{X}$ be set of all state trajectories.

History-based sensor mapping:

$$h : \tilde{X} \rightarrow Y$$

Preimages again:

$$h^{-1}(y) = \{\tilde{x} \in \tilde{X} \mid y = h(\tilde{x})\}$$

$h$ induces a partition of $\tilde{X}$.

A history-based sensor lattice is obtained over $\tilde{X}$.

Linear velocity of planar robot: $(v_x, v_y)$

LINEAR ODOMETER:

$$y = \theta_0 + \int_0^t \sqrt{v_x^2 + v_y^2} ds$$

$v_x$ and $v_y$ are part of the state.

For example, $x = (p_x, p_y, \theta, v_x, v_y)$

ANGULAR ODOMETER:

$$y = \theta_0 + \int_0^t \dot{\theta}(s)ds$$

# Delayed Measurement Sensor



Observe what the state was one second ago.

DELAYED MEASUREMENT SENSOR:

$$y = \begin{cases} \tilde{x}(t-1) & \text{if } t \geq 1 \\ \# & \text{otherwise} \end{cases}$$

$\#$ means no measurement yet available.

# Discrete-Time Odometer



Fixed time increment $\Delta t > 0$.

$\tilde{p}(t)$ is robot position in $\mathbb{R}^2$ at time $t$

DISCRETE-TIME ODOMETER:

$$h(\tilde{x}) = \sum_{i=1}^{\lceil t/\Delta t \rceil} \|\tilde{p}(i\Delta t) - \tilde{p}((i-1)\Delta t)\|$$

This this yields an estimate of the total distance travel.

It looks like a temporal filter, which is coming soon.

# Part 2 Summary



- Physical sensors and their characteristics
- Virtual sensors vs. physical sensors
- Families: Depth, detection, relational, gap, field
- Uncertainty comes from preimages!
- The sensor lattice
- Disturbances, history-based, state-time

To make better filters and planners, you need to find the appropriate virtual sensors for your task.